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Wavevector-Frequency Analysis With Applications to Acoustics

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Preface

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This document publishes, in preliminary form, the first 5 chapters of a 10-chapter monograph entitled "Wavevector-Frequency Analysis with Applications to Acoustics." This monograph is intended to be tutorial in nature, and is published in this preliminary form to elicit constructive comments and criticisms from the addressees of the initial distribution list.

The first chapter is introductory, and includes a brief history of the subject. Chapter 2 defines and interprets the parameters that describe harmonic waves, and illustrates how an arbitrary wave field is described in terms of these parameters. The third chapter introduces linear systems and their classifications, and treats the wavevector-frequency characteristics of space- and time-invariant systems. The fourth chapter describes the wavevector-frequency characteristics of space-varying, but

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time-invariant, linear systems, and the fifth chapter treats those characteristics of coupled linear systems.

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CHAPTER 1
INTRODUCTION

This monograph presents an approach to the description and analysis of acoustic fields and systems that parallels the approach developed in signal processing and linear systems theories to describe and analyze electrical and communications signals and systems. This approach, called wavevector-frequency analysis, is complementary to traditional methods of acoustics.

Wavevector-frequency analysis is simply the description of a space-time field or system in terms of the Fourier conjugates of the independent spatial and temporal variables of the field or system. The wavevector is the Fourier conjugate of the spatial vector variable and the frequency is the conjugate of the time variable.

The primary advantage of expressing the acoustic field in terms of wavevector and frequency, rather than space and time, is that the Fourier transformation often simplifies the mathematical description of the field or acoustic system, thereby facilitating mathematical analysis and physical interpretation.

The formalism of this approach to acoustics evolved over the course of an extended research effort to understand and characterize certain acoustic fields associated with turbulent flow. A brief history of this research effort will put the evolution of this approach in perspective.

1.1 HISTORY AND PERSPECTIVE

The pressure and vibrational fields associated with turbulent flow over various vehicles have been the subject of continuing research over the past 30 years. Turbulent flow is a random process in space and time. Therefore, the

turbulent pressure and turbulent flow-excited vibration fields are also random functions of space and time. The description, measurement, and analysis of these random fields requires knowledge of statistics and signal processing. The physics of turbulent flow, however, is the domain of the hydrodynamicist. The vibration of structural elements of the vehicles requires knowledge of elasticity and random vibrations. The acoustician desires descriptions and models of the pressure and vibrational fields to predict the noise in the near- and farfields of the vehicle. By necessity, therefore, the study of flow and flow-induced noise was an interdisciplinary effort.

As might be expected, the cooperative effort between researchers in these different specialties over a long period of time led to extensive cross-fertilization among disciplines. This cross-fertilization resulted in an entirely new set of specialties in the field of acoustics. The combinations of disciplines comprising the best known of these new specialties, hydro-acoustics and aeroacoustics, are evident from their names. Some combinations of disciplines, however, did not lead to such descriptive names nor to such widespread recognition. One such combination is the subject of this monograph.

The name wavevector-frequency analysis designates the specialized extension of traditional signal processing and linear systems theories developed, over the course of this research effort, to describe and analyze the turbulent pressure and turbulent flow-excited vibration fields. Traditional signal processing and linear system theories were developed for electrical and communication systems problems, in which the fields of interest are functions of the single independent variable, time. In linear systems theory, it is demonstrated that Fourier transformation of these temporal fields into corresponding fields in frequency (the Fourier conjugate of the time variable) often simplifies the mathematical description of the system and, thereby, facilitates physical prediction or interpretation of the system. To exploit these potential advantages for empirical investigations, the signal processor developed techniques to determine or measure the frequency characteristics of random and deterministic time fields. To describe and analyze the space-time fields associated with turbulent flow and flow-excited vibrations, it was necessary to extend these traditional theories

to include the spatial coordinates, or spatial position vector, as an additional independent field variable.

The advantages of Fourier transform techniques for analysis and interpretation of linear systems had long been recognized by researchers in other branches of physics, including those in acoustics. The pressure and velocity fields encountered in acoustics are functions of both space and time. Therefore, to obtain Fourier-transformed descriptions of acoustic fields, it was necessary to define a Fourier conjugate of the spatial position vector as well as the conjugate of the time variable (frequency). The Fourier conjugate of the spatial vector was defined as the wavevector. Wavevector-frequency descriptions of acoustic fields were traditionally employed only as an intermediate step in the theoretical prediction of the space-time characteristics of these fields. Nonetheless, these theoretical studies provided a solid basis for the formalized extension of traditional linear systems theory to space-time fields. However, because of the introduction of the additional vector variable, the signal processor's task of developing the theory and means of measuring the wavevector-frequency characteristics of space-time fields was a difficult one.

In 1967, Maidanik and Jorgensen[1] proposed a method for direct measurement of the wavevector-frequency spectrum of the wall pressure fluctuations in a turbulent boundary layer. In 1971, Blake and Chase[2] used this technique to perform such measurements. The demonstrated ability to measure the wavevector-frequency characteristics of space-time fields promoted wavevector-frequency analysis from an interesting, and sometimes useful, mathematical technique to a potentially powerful tool for interpretation and analysis of experimental data. It therefore prompted increased research efforts to develop new techniques for wavevector-frequency measurement and analysis. These research efforts continue today in a wide variety of scientific fields dealing with space-time fields.

1.2 MOTIVATION AND OBJECTIVE

Despite the demonstrated utility of wavevector-frequency analysis for characterization and analysis of the acoustic fields generated by turbulent

flow over vehicles, the application of wavevector analysis techniques in acoustics has been left to the specialist. The motivation for this monograph is to encourage a wider understanding and application of these powerful techniques.

One impediment to the adoption of these techniques by the nonspecialist is the lack of a comprehensive tutorial treatment of wavevector-frequency analysis. Although the existing theoretical and experimental capabilities in this specialty resulted from research conducted in universities, government laboratories, and private industry, there has been no attempt to organize the disparate theoretical and experimental results of these research efforts into a single comprehensive reference source.

The objective of this monograph is to provide a tutorial treatment of wavevector-frequency analysis and its application to acoustics.

Because the primary purpose of this work is to teach the fundamentals of wavevector-frequency analysis for acoustic applications, the monograph will not provide an exhaustive review of the manifold publications of theoretical and experimental research in this field. Rather, it will meld selected results from those references into a format that mathematically defines, physically interprets, and (where possible) illustrates by experimental data, the essential aspects of wavevector-frequency analysis. It is expected that readers interested in a particular topic will use the references cited at the end of each chapter to expand their sources of information.

1.3 ORGANIZATION

The 10 chapters comprising this text treat 5 topics. Basic definitions and relationships are presented in chapter 2. Chapters 3, 4, and 5 present linear systems theory for space- and time-invariant, space-varying, and coupled systems, respectively. Chapters 6 and 7 treat the description of random space-time fields and the response of linear systems to such random fields. The problems of measurement and estimation of wavevector-frequency spectra are treated in chapters 8 and 9. Chapter 10 presents some illustrative examples of the use of wavevector-frequency analysis techniques.

1.4 DEPTH AND PREREQUISITES

This monograph is intended as a tutorial source for practicing scientists and engineers. The level of the material presented is equivalent to that encountered in a first-year graduate course. The reader is assumed to have a basic understanding of acoustics (including vibrations), Fourier transforms and series, simple generalized functions, and statistics. Although many of the fundamental concepts in these subjects are reviewed in this book, their treatment is not rigorous. Rather, such reviews are somewhat cursory and are only included to reacquaint the reader with certain fundamentals or to improve the continuity of certain arguments. For rigorous treatments of such fundamental concepts, the reader should consult standard references on the appropriate subject.

1.5 REFERENCES

1. G. Maidanik and D. W. Jorgensen, "Boundary Wave-Vector Filters for the Study of the Pressure Field in a Turbulent Boundary Layer," Journal of the Acoustical Society of America, vol. 42, no. 2, August 1967, pp. 494-501.

2. W. K. Blake and D. M. Chase, "Wavenumber-Frequency Spectra of Turbulent Boundary Layer Pressure Measured by Microphone Arrays," Journal of the Acoustical Society of America, vol. 49, no. 3, March 1971, pp. 862-877.

CHAPTER 2

WAVES AND THEIR DESCRIPTORS

This chapter defines, and physically interprets, the parameters used to describe the spatial and temporal characteristics of harmonic waves. The mathematical description of an arbitrary wave in terms of these parameters is developed.

Van Nostrand's Scientific Encyclopedia[1] defines a wave as a "disturbance which is propagated in a medium in such a manner that at any point in the medium the displacement is a function of the time, while at any instant the displacement at a point is a function of the position of the point." This general definition establishes that waves are fields in which the variation of some physical quantity is specified over some region of space and time. Note that this definition does not require any specific form of the temporal or spatial variation of the disturbance. However, the word "propagated" implies some relationship between the spatial and temporal variables. From the above, it is clear that a wave is not a specific space-time field. Rather, a wave is any member of a class of space-time fields that describes a disturbance which propagates in space and time.

One of the simplest waves is the harmonic wave. A harmonic wave is defined as one in which the disturbance varies sinusoidally in space and time. We begin our study and characterization of waves with a specific form of the harmonic wave called the plane harmonic wave.

2.1 THE PLANE HARMONIC WAVE

Consider a disturbance, say the pressure in a fluid, described by

$$p(\vec{x},t) = P \exp[i(\vec{k}\cdot\vec{x} + \omega t)] \quad . \tag{2-1}$$

Here, p denotes the pressure, $\vec{x} = [x_1,x_2,x_3]$ is the spatial position vector, t designates time, P is a complex constant that represents the

amplitude of the disturbance, $\vec{k} = [k_1,k_2,k_3]$ is the (constant) wave vector, ω is the (constant) circular frequency, exp z denotes e^z, $\vec{k}\cdot\vec{x}$ denotes the inner (or dot) product of the vectors $\vec{k}$ and $\vec{x}$, and i is the square root of minus one.

The physical pressure is, of course, a real function of space and time. Therefore, when expressing the pressure in the complex form of equation (2-1), we mean that the physical pressure, $p_r(\vec{x},t)$, is the real part of the complex function. That is, we mean

$$p_r(\vec{x},t) = P_{mag} \cos(\vec{k}\cdot\vec{x} + \omega t + \phi) \ , \tag{2-2}$$

where P_{mag} and ϕ are the absolute value and argument of the complex amplitude, P, respectively.

The argument of the cosine in equation (2-2), that is, $\vec{k}\cdot\vec{x} + \omega t + \phi$, is defined as the phase of the harmonic wave, and P_{mag} is the amplitude of the physical wave. Therefore, the effect of varying the spatial or temporal coordinates is simply to change *the phase of the wave. Note that* ϕ *is the* phase of the wave at $\vec{x} = [0,0,0]$ and $t = 0$.

The period of a harmonic wave is defined as the time difference, T, between successive repetitions, or cycles, of the wave at a fixed point in space. Recall that, in equations (2-1) and (2-2), the wavevector and frequency are constants. Thus, at any fixed point in space, say $\vec{x}_0$, the pressure varies only with time. That is,

$$p(\vec{x}_0,t) = P_{mag} \exp[i(\omega t + \vec{k}\cdot\vec{x}_0 + \phi)] \ . \tag{2-3}$$

Mathematically, the period is defined as the smallest positive value of T for which

$$p(\vec{x}_0,t + T) = p(\vec{x}_0,t) \tag{2-4}$$

for all t. By equations (2-3) and (2-4), it is clear that the period corresponds to the time increment required to increase the phase of the wave

by 2π radians, and it is given by

$$T = 2\pi/\omega \ . \tag{2-5}$$

It follows, then, that the circular frequency, ω, is related to the period by

$$\omega = 2\pi/T \tag{2-6}$$

and is the time rate of change of phase of the wave in radians per second.

A more familiar definition of frequency is the temporal frequency, f, which is defined as the number of repetitions, or cycles, of the wave per unit time. One cycle corresponds to a phase change of 2π radians; thus, the temporal and circular frequencies are related by

$$f = \omega/(2\pi) = 1/T \ . \tag{2-7}$$

By the above, it is evident that both circular and temporal frequency define the time rate of repetition of the wave. However, this rate is measured in *different units*. That is, circular frequency is the time rate of change of phase, where a phase change of 2π radians is required for one repetition. Temporal frequency, on the other hand, is the number of repetitions of the wave per unit time.

An important concept in the description and characterization of harmonic waves is that of the phase front. A phase front is defined as a surface in space over which the phase of the wave is constant. According to equations (2-1) and (2-2), when the phase is constant, the value of the pressure, $p(\vec{x},t)$, is constant. Thus, a phase front corresponds to a surface of constant pressure in space associated with a particular phase of the wave.

Consider the phase front of the wave defined by equation (2-2) associated with the constant phase, β. The phase front is then defined by

$$\vec{k}\cdot\vec{x} + \omega t + \phi = \beta \tag{2-8}$$

and is designated as phase front β. At time $t = t_0$, the surface of constant

phase associated with phase front ß is given by

$$\vec{k}\cdot\vec{x} + \omega t_0 + \phi = \beta \tag{2-9}$$

and is a function of $\vec{x}$ only. As $\vec{k}$ is a constant, equation (2-9) defines a plane in the three-dimensional space, $\vec{x}$. The constant, ß, is arbitrary. Therefore, all phase fronts of the harmonic wave defined by equations (2-1) and (2-2) are planes. Consequently, waves having the mathematical form of equations (2-1) and (2-2) are called plane harmonic waves, and their phase fronts are often referred to as phase planes.

Let $\vec{x}_A$ and $\vec{x}_B$ be vectors defining two points, A and B, in the phase plane specified by equation (2-9), as illustrated in figure 2-1.

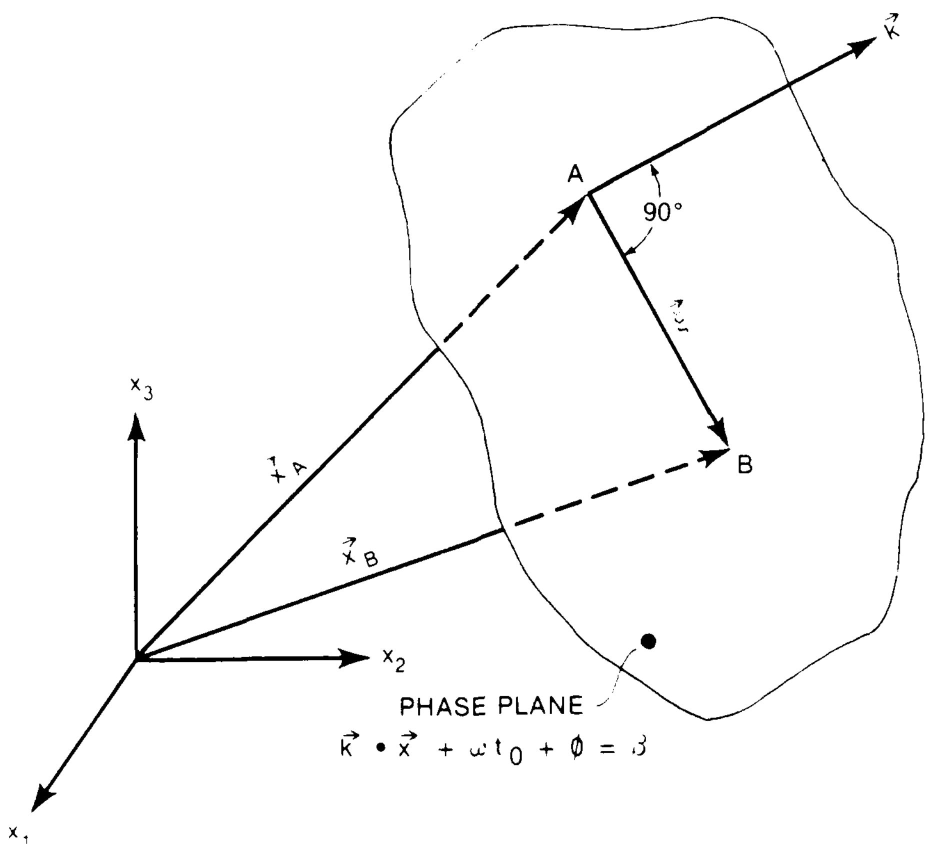

Figure 2-1. Relative Geometry Between the Wavevector and a Line in the Phase Plane

The nonzero vector $\vec{\xi} = \vec{x}_B - \vec{x}_A$ then defines a vector in the phase plane. As both $\vec{x}_A$ and $\vec{x}_B$ satisfy equation (2-9), it follows that

$$\vec{k} \cdot \vec{\xi} = 0 \ . \tag{2-10}$$

Neither $\vec{k}$ nor $\vec{\xi}$ is a zero vector. It follows, then, that the wavevector $\vec{k}$ is perpendicular to the vector $\vec{\xi}$. However, as the vectors $\vec{x}_A$ and $\vec{x}_B$ are arbitrary, the vector $\vec{\xi}$ is also arbitrary. Therefore, the wavevector, $\vec{k}$, is perpendicular to every vector in the phase plane. Consequently, the wavevector $\vec{k}$ is perpendicular to the plane of the phase front. Furthermore, as $\vec{k}$ is a constant vector, it is clear that all phase planes of our harmonic wave are parallel.

By taking the time derivative of equation (2-8), we obtain (for constant β)

$$\vec{k} \cdot \vec{v} + \omega = 0 \ , \tag{2-11}$$

where $\vec{v}$ is the velocity vector of the phase front at any point $\vec{x}$ and is defined by

$$\vec{v} = \frac{d\vec{x}}{dt} \ . \tag{2-12}$$

It is evident that there are an infinite number of velocities which satisfy equation (2-11). However, as $\vec{k}$ is a constant, all possible velocity vectors defined by equation (2-11) must have the same component normal to the phase front: that is, in the direction parallel to $\vec{k}$. If we denote the velocity normal to the phase front by $\vec{c}_p$, it follows from equation (2-11) that

$$\vec{k} \cdot \vec{v} = \vec{k} \cdot \vec{c}_p = k c_p \cos \Theta = -\omega \ , \tag{2-13}$$

where k and c_p denote the magnitudes of the corresponding vectors and Θ is the angle between the vectors $\vec{k}$ and $\vec{c}_p$. As $\vec{k}$ and $\vec{c}_p$ are parallel vectors, it follows from equation (2-13) that $\Theta = \pi$ for $\omega > 0$ and $\Theta = 0$ for $\omega < 0$. Thus, the wavevector, $\vec{k}$, is directed opposite to the normal velocity of the phase front for $\omega > 0$ and coincident to the normal velocity of the phase front for $\omega < 0$. Further, as $\vec{k}$ and ω are constants, the normal velocity, $\vec{c}_p$, is a constant and is given by

$$\vec{c}_p = \frac{-\vec{k}\omega}{k^2} . \tag{2-14}$$

The velocity $\vec{c}_p$ is defined as the phase velocity of the wave and is the apparent velocity at which the planar phase front propagates through the medium. The word "apparent" is used because only the normal component of the velocity of the phase plane effects an apparent change in the spatial location of the phase plane; the tangential components produce only in-plane slippage of the front. For a more comprehensive treatment of the kinematics of wavefronts, the reader is directed to reference 2.

For a known phase velocity of the plane wave, the wavevector is easily determined by

$$\vec{k} \approx \frac{-\vec{c}_p\omega}{c_p^2} . \tag{2-15}$$

Returning our attention to equations (2-1) and (2-2), it is a simple matter to show that, at any fixed time, the pressure field is periodic in space. That is, a spatial vector $\vec{\xi}$ exists such that, at the fixed time t_0,

$$p(\vec{x} + \vec{\xi},t_0) = p(\vec{x},t_0) \tag{2-16}$$

for all $\vec{x}$. The nonzero vectors, $\vec{\xi}$, that satisfy equation (2-16) are easily shown to be those for which

$$\vec{k}\cdot\vec{\xi} = 2n\pi , \quad n = \pm 1,\ 2,\ 3,\ \ldots . \tag{2-17}$$

The allowable values of $\vec{\xi}$ in equation (2-17) correspond to all vector separations between a reference phase plane and a series of other phase planes. If the phase of the reference plane is β, the phases associated with the other planes are $\beta + 2n\pi$.

The wavelength (or spatial period) is defined as the distance, measured normal to the phase plane, between successive spatial repetitions of the wave. If we denote the wavelength by λ and recall that the vector $\vec{k}$ is normal

to the wavefront, it follows from equation (2-17) that

$$\lambda = 2\pi/k \ . \tag{2-18}$$

Thus, for a known wavelength, the magnitude of the wavevector is given by

$$k = 2\pi/\lambda \ . \tag{2-19}$$

As λ is the minimum distance between successive repetitions of the wave and is measured parallel to $\vec{k}$, it follows from equation (2-19) that the wavevector can be interpreted as the magnitude and direction of the maximum rate of change of phase in space. This same interpretation of the wavevector can be obtained by considering the gradient of the phase. It is straightforward to show that the gradient of the phase, which by definition corresponds to the maximum spatial rate of change of phase, is equal to $\vec{k}$.

By returning our attention to equation (2-17), it is apparent that, if the vector $\vec{\xi}$ is taken parallel to the x_1 axis, the distance between successive repetitions of the wave is

$$\vec{\xi}_1 = 2\pi/k_1 \ . \tag{2-20}$$

This distance is defined as the projected wavelength in the x_1 coordinate direction and is designated by λ_1. Similarly, projected wavelength components may be defined in the other two coordinate directions, resulting in the relationships

$$\lambda_1 = 2\pi/k_1 \ , \quad \lambda_2 = 2\pi/k_2 \ , \quad \lambda_3 = 2\pi/k_3 \ . \tag{2-21}$$

The magnitude of the three components of the wavevector can therefore be expressed in terms of the projected wavelength components by

$$k_1 = 2\pi/\lambda_1 \ , \ k_2 = 2\pi/\lambda_2 \ , \ k_3 = 2\pi/\lambda_3 \tag{2-22}$$

and can be interpreted as the spatial rate of change of phase in the respective coordinate directions.

Some texts (see reference 3, for example) define the wavenumber components in terms of the number of cycles of the wave per unit length (which corresponds to $k_j = 1/\lambda_j$, j = 1, 2, 3) rather than the spatial rate of change of phase implied by equation (2-22). In this text, the wavevector will always be defined such that its components are consistent with equation (2-22). However, if one desires to express the wavenumber components in terms of cycles per unit length, the conversion is easily made by using arguments similar to those leading to equation (2-7).

2.2 MATHEMATICAL REVIEW

This section reviews some of the mathematical concepts and techniques that will be used in the course of this text. The review is included only for the purpose of reacquainting the reader with these concepts and techniques and for establishing certain conventions that we shall follow throughout this text. Therefore, this review will be conducted without any pretense of rigor, and the reader is encouraged to consult standard texts, as necessary, to supplement the material presented here.

2.2.1 Fourier Transforms

The Fourier integral theorem states that a function g(t) can be represented as an integral of its harmonic elements.[4] That is,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp(i\omega t)\, d\omega \ , \tag{2-23}$$

where $G(\omega)$ is the complex amplitude (within a factor of 2π) of each harmonic element and is given by

$$G(\omega) = \int_{-\infty}^{\infty} g(t) \exp(-i\omega t)\, dt \ . \tag{2-24}$$

The functions g(t) and $G(\omega)$ constitute a Fourier transform pair, and the variable ω is called the Fourier conjugate of the variable t. The placement

of the factor of 2π in the definition of the transform pair is, within certain constraints, arbitrary. The choice in equations (2-23) and (2-24) is consistent with the convention used by electrical engineers.

To paraphrase Lighthill,[5] considerable literature has been devoted to determining the conditions on g(t) sufficient for equations (2-23) and (2-24) to be valid representations. For the fields, g(t), treated in this text, equations (2-23) and (2-24) are valid representations.

Sneddon[6] shows that the theory of Fourier transforms of functions of a single variable can be extended to functions of several variables. Thus, wave fields, which are functions of space and time, may be represented in terms of multidimensional Fourier transforms. For example, let $p(\vec{x},t)$ denote the space-time field associated with a pressure wave. Then, $p(\vec{x},t)$ can be represented by

$$p(\vec{x},t) = \frac{1}{(2\pi)^4} \iint_{-\infty}^{\infty} P(\vec{k},\omega)\exp[i(\vec{k}\cdot\vec{x} + \omega t)]\, d\vec{k}\, d\omega \,, \tag{2-25}$$

where

$$P(\vec{k},\omega) = \iint_{-\infty}^{\infty} p(\vec{x},t)\exp[-i(\vec{k}\cdot\vec{x} + \omega t)]\, d\vec{x}\, dt \,. \tag{2-26}$$

In equations (2-25) and (2-26), $d\vec{x}$ denotes $dx_1dx_2dx_3$ and $d\vec{k}$ denotes $dk_1dk_2dk_3$. The Fourier conjugate of the spatial vector variable, $\vec{x}$, is the wavevector, $\vec{k}$, and the Fourier conjugate of the time variable, t, is the circular frequency, ω.

Note that the integrand of equation (2-25) is a harmonic plane wave of the form of equation (2-1), with a complex amplitude of $P(\vec{k},\omega)$. Thus, it is evident, by equation (2-25), that expressing the pressure field as a Fourier transform is equivalent to representing that field as a summation, or superposition, of harmonic plane waves, where each harmonic component is characterized by a distinct wavevector and frequency. The wavevector-

frequency transform, $P(\vec{k},\omega)$, of the space-time field, $p(\vec{x},t)$, can be interpreted as the relative complex amplitude of each harmonic plane wave component comprising the pressure field.

It should be emphasized that the components of the wavevector, $\vec{k}$, and the circular frequency, ω, are real variables. If we require the wave field variable (e.g., the pressure in equation (2-25)) to be real also, then we require that

$$p(\vec{x},t) = p^{*}(\vec{x},t) , \tag{2-27}$$

where the asterisk denotes the complex conjugate. It follows from equations (2-25) and (2-27) that, for the pressure to be real,

$$P(\vec{k},\omega) = P^{*}(-\vec{k},-\omega) . \tag{2-28}$$

Thus, the wavevector-frequency transform of a real space-time field has conjugate symmetry in both wavevector and frequency.

2.2.2 Generalized Functions

Many of the operations involving Fourier transforms are facilitated by the use of generalized functions. Further, in some of the chapters to follow, generalized functions will be used for either notational or mathematical convenience. Therefore, before proceeding with further discussion of the Fourier transform and its properties, it is convenient to introduce the three generalized functions that will be used repeatedly throughout this text. For a more rigorous treatment of these generalized functions, the reader is referred to such texts as Lighthill[5] or Papoulis.[7]

The generalized function used most often is the Dirac delta function, denoted by δ. The delta function is defined by[8]

$$\delta(t - t_0) = 0 , \quad \text{for } t \neq t_0 , \tag{2-29}$$

and is sufficiently large in the vicinity of $t = t_0$ that

$$\int_{t_1}^{t_2} \delta(t - t_0)\, dt = 1 , \qquad (2\text{-}30)$$

where $t_1 < t_0 < t_2$. The delta function has the integral property that

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0)\, dt = g(t_0) , \qquad (2\text{-}31)$$

where g(t) is any function of t that is continuous at t_0. By equation (2-31), it is seen that the delta function can be used to sample a function at any discrete argument of that function.

The second generalized function we shall use is the Heaviside, or unit step, function. The Heaviside function, denoted by U, is the indefinite integral of the Dirac delta function and is defined by the discontinuous function

$$U(t - t_0) = \int_{-\infty}^{t} \delta(y - t_0)\, dy = \begin{cases} 0 , & t < t_0 , \\ 1 , & t > t_0 . \end{cases} \qquad (2\text{-}32)$$

The Heaviside function has the integral property that

$$\int_{-\infty}^{\infty} U(t - t_0)g(t)\, dt = \int_{t_0}^{\infty} g(t)\, dt , \qquad (2\text{-}33)$$

where g(t) is any function that is continuous at $t = t_0$.

The last of the generalized functions we require are the derivatives of the delta function. The n^{th} derivative of the delta function is denoted by

$$\delta^{(n)}(t - t_0) = \frac{d^n \delta(t - t_0)}{dt^n} . \qquad (2\text{-}34)$$

The derivatives of the delta function have the property that, for any good function, g(t),

$$\int_{t_1}^{t_2} \delta^{(n)}(t - t_0)g(t)\,dt = (-1)^n \frac{d^n g(t_0)}{dt^n} , \qquad (2\text{-}35)$$

where $t_1 < t_0 < t_2$. Lighthill[9] defines a "good" function as one that is everywhere differentiable any number of times and is such that it and all its derivatives are, at most, of order $|x|^{-N}$ as $|x|$ approaches infinity for all N. Thus, a good function decays, for large $|x|$, faster than any inverse power of $|x|$.

2.2.3 Some Useful Relationships and Interpretations

By use of the generalized functions and the Fourier transform, we may deduce some relationships that will be of use in forthcoming chapters. Further, some of these mathematical relationships can be interpreted in terms of the composition and characterization of wave fields.

One especially useful relationship is the Fourier transform of the delta function. If, in equation (2-24), we set

$$g(t) = \delta(t - t_0) , \qquad (2\text{-}36)$$

then, by equation (2-31), it follows that

$$G(\omega) = \exp(-i\omega t_0) . \qquad (2\text{-}37)$$

From equations (2-23), (2-36), and (2-37), it follows that

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - t_0)]\,d\omega . \qquad (2\text{-}38)$$

A similar relation can be developed for $\delta(\omega - \omega_0)$.

Consider now a wave field (for consistency, we will again use the pressure field, $p(\vec{x},t)$) that has the Fourier transform

$$P(\vec{k},\omega) = (2\pi)^4 P_0 \delta(\vec{k} - \vec{k}_0)\delta(\omega - \omega_0) \ , \tag{2-39}$$

where

$$\delta(\vec{k} - \vec{k}_0) = \delta(k_1 - k_{01})\delta(k_2 - k_{02})\delta(k_3 - k_{03}) \tag{2-40}$$

and P_0 is a complex constant. By equations (2-25), (2-31), (2-39), and (2-40), it is easily shown that the pressure field resulting from the wavevector-frequency transform of equation (2-39) is

$$p(\vec{x},t) = P_0 \exp[i(\vec{k}_0 \cdot \vec{x} + \omega_0 t)] \ . \tag{2-41}$$

Equations (2-39) and (2-41) constitute a four-dimensional Fourier transform pair.

Comparison of equation (2-41) with equation (2-1) reveals that the pressure field of equation (2-41) is a plane harmonic wave field characterized by the single wavevector $\vec{k}_0$, the single frequency ω_0, and the complex amplitude P_0. This example shows distinctly one of the potential advantages of expressing a wave field in terms of its wavevector-frequency transform. That is, in this example, the Fourier transform maps the plane harmonic wave field, which exists over all space and time, into a field in the Fourier conjugate, or wavevector-frequency, domain, which is nonzero only at a single point.

A particularly useful property of the Fourier transform is the property of superposition. That is, if g(t) in equation (2-24) is given by

$$g(t) = \sum_{n=1}^{N} g_n(t) \ , \tag{2-42}$$

then it follows that

$$G(\omega) = \sum_{n=1}^{N} G_n(\omega) \ . \qquad (2\text{-}43)$$

An interesting example of superposition in four dimensions can be illustrated by considering the pressure field resulting from the wavevector-frequency spectrum given by

$$P(\vec{k},\omega) = \frac{(2\pi)^4}{2} P_0[\delta(\vec{k} - \vec{k}_0)\delta(\omega - \omega_0) + \delta(\vec{k} + \vec{k}_0)\delta(\omega + \omega_0)] \ , \qquad (2\text{-}44)$$

where P_0 is a real constant. By equations (2-25) and (2-44), the pressure field is

$$p(\vec{x},t) = P_0 \cos(\vec{k}_0 \cdot \vec{x} + \omega_0 t) \ . \qquad (2\text{-}45)$$

Note that the pressure field of equation (2-45) is of the form of equation (2-2), with ø equal to zero. In this case, the pressure field is seen to be real, and its wavevector-frequency transform consists of two discrete components. Note that the Fourier transform of the real-valued pressure field (equation (2-44)) satisfies the condition of equation (2-28).

At the risk of belaboring a point, we may use the principle of superposition to substantiate our physical interpretation of the Fourier transform of equation (2-25). That is, we have shown that wavevector-frequency transform comprised of the product of delta functions of the form of equation (2-39) produces a plane harmonic wave in the space-time domain of the form of equation (2-41). By the principle of superposition, then, a transform comprised of a summation of many different products of delta functions will produce a space-time field comprised of a summation of the corresponding plane harmonic waves. Therefore, if we write the transform of the pressure field in the form of the weighted superposition of products of delta functions, that is,

$$P(\vec{k},\omega) = \iint_{-\infty}^{\infty} P(\vec{\mu},\Omega)\delta(\vec{k} - \vec{\mu})\delta(\omega - \Omega)\ d\vec{\mu}\ d\Omega \ , \qquad (2\text{-}46)$$

then the resulting pressure field is a weighted superposition of plane harmonic waves. By equations (2-25) and (2-31), that weighted superposition of plane waves is given by

$$p(\vec{x},t) = (2\pi)^{-4} \iint_{-\infty}^{\infty} P(\vec{\mu},\Omega)\exp[i(\vec{\mu}\cdot\vec{x} + \Omega t)]\, d\vec{\mu}\, d\Omega \ , \tag{2-47}$$

which is merely a restatement of the multidimensional Fourier transform of equation (2-25). Therefore, the expression of a wave field as the multidimensional Fourier transform of equation (2-25) can be physically interpreted as the representation of that wave field as a superposition of plane harmonic waves. The wavevector-frequency transform of the space-time wave field represents the relative amplitudes and phases of these various harmonic wave components.

In the forthcoming chapters, we will often require the Fourier transforms of temporal or spatial derivatives of fields. Let g(t) be given by equation (2-23) and define

$$f(t) = \frac{d^n g(t)}{dt^n} \ . \tag{2-48}$$

If $F(\omega)$ denotes the Fourier transform of f(t), it is straightforward to show, by equations (2-23) and (2-48), that

$$F(\omega) = (i\omega)^n G(\omega) \ . \tag{2-49}$$

By similar arguments, it may be shown that the inverse Fourier transform of the n^{th} derivative of $G(\omega)$ with respect to ω is equal to $(-it)^n g(t)$.

An interesting and useful application of equations (2-48) and (2-49) is illustrated by the following example. Consider the equation

$$(\omega^2 - \omega_0^2)G(\omega) = 0 \ , \tag{2-50}$$

where we wish to determine $G(\omega)$. By performing the inverse Fourier transform of equation (2-50) and by utilizing equation (2-48), equation (2-49), and the principle of superposition, we may show that

$$g(t) = A \exp(i\omega_0 t) + B \exp(-\omega_0 t) \ , \tag{2-51}$$

where A and B are constants. By use of equations (2-24) and (2-38), it follows that the $G(\omega)$ satisfying equation (2-50) is given by

$$G(\omega) = A\delta(\omega - \omega_0) + B\delta(\omega + \omega_0) \ . \tag{2-52}$$

This result will prove useful in forthcoming chapters.

As a final mathematical note, consider the Fourier transform of the product of two functions, $g(t)$ and $f(t)$. If the transforms of g and f are denoted by G and F, we may use equations (2-23) and (2-38) to show that

$$\int_{-\infty}^{\infty} g(t)f(t)\exp(-i\omega t)\, dt = (2\pi)^{-1} \int_{-\infty}^{\infty} G(\omega - \Omega)F(\Omega)\, d\Omega \ . \tag{2-53}$$

The integral on the right-hand side is called the convolution of the functions G and F. By equation (2-53), it is seen that the Fourier transform of a product is the convolution of the transforms of the functions making up the product. By similar arguments, it may be shown that the inverse transform of a product also results in a convolution. That is,

$$(2\pi)^{-1} \int_{-\infty}^{\infty} G(\omega)F(\omega)\exp(i\omega t)\, d\omega = \int_{-\infty}^{\infty} f(t - \Theta)g(\Theta)\, d\Theta \ . \tag{2-54}$$

2.3 WAVEVECTOR-FREQUENCY DESCRIPTION OF WAVE FIELDS

This section reviews certain physical and mathematical concepts presented in the first two sections to clarify the rationale for describing wave fields in terms of their wavevector-frequency characteristics. In addition,

wavevector-frequency descriptions of wave fields in one and two spatial dimensions will be presented and discussed.

2.3.1 Review and Perspective

The justification for the description of wave fields as a function of wavevector and frequency, rather than space and time, is found in the multidimensional Fourier transform pair of equations (2-25) and (2-26). By these transform relationships, it must be concluded that $P(\vec{k},\omega)$ and $p(\vec{x},t)$ constitute equivalent descriptions of the wave field, inasmuch as either description may be derived from the other via the appropriate Fourier transformations. Therefore, a description of a wave field as a function of wavevector and frequency is as valid and complete as the description of that field as a function of space and time.

The reader might ask, with some justification, why the kinematics of the plane harmonic wave, rather than the simple argument presented above, was the focus of the initial section of this chapter. The reason for this choice was that, in my experience, the primary impediment to the understanding of the wavevector-frequency descriptions of fields is not the concept of the Fourier transform: it is the concept of the wavevector. The scientist or engineer has no problem envisioning the pressure field, $p(\vec{x},t)$, because space and time are familiar physical concepts. Envisioning the field $P(\vec{k},\omega)$, on the other hand, is likely to prove difficult because the wavevector is an unfamiliar physical concept. The concept of frequency, however, is well understood by acousticians. For this reason, the primary emphasis in this chapter was to define and physically interpret the wavevector.

The definition and interpretation of the wavevector were addressed by studying the kinematics of a plane harmonic wave, which corresponds to a wavevector-frequency field, $P(\vec{k},\omega)$, containing a single discrete wavevector-frequency component. It was shown that the space-time field associated with a plane harmonic wave is completely determined by the amplitude and phase of the wave. The amplitude specifies only the magnitude (and, in the case of a complex wave, the initial phase) of the disturbance associated with the wave.

All information regarding the spatial and temporal variation of the wave is contained in the phase.

As was shown in equation (2-8), the phase is a linear function of the components of the spatial position vector, $\vec{x}$, and time, t. The constants of proportionality are the components of the wavevector, $\vec{k}$, and the circular frequency, ω. The wavevector components define the rate of spatial repetition of the wave in each of the corresponding spatial coordinate directions at any fixed time. The circular frequency defines the rate of temporal repetition of the wave at any fixed point in space. The direction and speed of propagation of the plane harmonic wave are determined by appropriate combinations of the wavevector and circular frequency.

By equation (2-1), it is evident that knowledge of the (complex) amplitude, the wavevector, and the circular frequency is sufficient to define the field of the plane harmonic wave over all space and time. Further, it was shown in section 2.2.3 that expressing a space-time wave field as the multiple Fourier transform of equation (2-25) is equivalent to representing that field as a superposition of plane harmonic waves. Therefore, if one knows the complex amplitude, the wavevector, and the circular frequency of each plane harmonic wave comprising that superposition, the wave field can be uniquely defined over all space and time. This is precisely the information provided by the wavevector-frequency description (or transform) of the wave field, which is denoted by $P(\vec{k},\omega)$ in equations (2-25) and (2-26).

By the arguments presented above, the wavevector-frequency description (or transform) of a wave field specifies the complex amplitudes of all harmonic plane waves comprising that field as a function of the rates of spatial repetition (in each coordinate direction) and temporal repetition corresponding to each harmonic wave component.

2.3.2 Wave Fields in One and Two Spatial Dimensions

For generality, the wave fields treated thus far in this chapter have been assumed to have three-dimensional spatial variation. In forthcoming chapters, many of the illustrative examples will treat wave fields with spatial

variation in only one or two coordinate directions. These one- and two-dimensional spatial fields are interpreted as special cases of the three-dimensional field below.

Consider the pressure field, $p(\vec{x},t)$, having the wavevector-frequency transform

$$P(\vec{k},\omega) = 2\pi\tilde{P}(\underline{k},\omega)\delta(k_3) \quad , \tag{2-55}$$

where $\underline{k}$ denotes the two-dimensional wavevector (k_1,k_2). By equation (2-25), the space-time field corresponding to equation (2-55) is given by

$$p(\vec{x},t) = \tilde{p}(\underline{x},t) = (2\pi)^{-3} \int_{-\infty}^{\infty} \tilde{P}(\underline{k},\omega)\exp[i(\underline{k}\cdot\underline{x} + \omega t)]\, d\underline{k}\, d\omega \quad , \tag{2-56}$$

where $\underline{x}$ denotes the two-dimensional spatial vector (x_1,x_2). By equation (2-56), it is evident that $\tilde{p}(\underline{x},t)$ is a pressure field that depends only on the two-dimensional spatial vector, $\underline{x}$, and the time, t. Further, by equations (2-55), (2-56), and (2-26), one can easily demonstrate that

$$\tilde{P}(\underline{k},\omega) = \int_{-\infty}^{\infty} \tilde{p}(\underline{x},t)\exp[-i(\underline{k}\cdot\underline{x} + \omega t)]\, d\underline{x}\, dt \quad . \tag{2-57}$$

Thus, $\tilde{P}(\underline{k},\omega)$ is the three-dimensional Fourier transform of $\tilde{p}(\underline{x},t)$ and is a function of the two-dimensional wavevector, $\underline{k}$, and the circular frequency, ω.

The characteristics of the wave field in two spatial dimensions can be interpreted as a special case of the three-dimensional spatial field. Equation (2-55) describes a wavevector-frequency field in which only plane harmonic waves having a zero spatial repetition rate in the x_3 coordinate direction (i.e., $k_3 = 0$) contribute to the space-time field. The physical interpretation of a zero spatial repetition rate of a plane harmonic wave in one coordinate direction is that there is no spatial variation of the wave in that coordinate direction. This is borne out by equation (2-56), which shows the resultant space-time field to be independent of the x_3 spatial coordinate.

As the space-time dependence of a plane harmonic wave is contained in the phase of the wave, it follows that the phase of each plane harmonic wave contribution to the wave field must also be independent of x_3. This conclusion is again supported by the form of the integrand of equation (2-56). If we consider a single wavevector-frequency component of the integrand, say $\underline{k}_0 = (k_{01}, k_{02})$ and ω_0, and denote the initial phase of that component by $\phi(\underline{k}_0, \omega_0)$, the phase front associated with the single wavevector component is given by

$$\underline{k}_0 \cdot \underline{x} + \omega_0 t + \phi(\underline{k}_0, \omega_0) = \beta \; . \qquad (2\text{-}58)$$

The phase front defined by equation (2-58) is a straight line in the (x_1, x_2) plane. This straight line can be interpreted as a special case of the phase plane defined by equation (2-9). Recall that the wavevector, $\vec{k}$, is perpendicular to the phase plane. The x_3 axis, in our Cartesian coordinate system, is perpendicular to the (x_1, x_2) plane. By equations (2-55) and (2-56), only plane harmonic wave components characterized by wavevectors $\vec{k} = (\underline{k}, 0)$ contribute to the space-time pressure field. For such wavevectors, it is easily shown that if $\vec{x}_3 = (0, 0, x_3)$, then $\vec{k} \cdot \vec{x}_3 = 0$. As the magnitudes of neither $\vec{k}$ nor $\vec{x}_3$ are, in general, zero, it follows that the phase planes are perpendicular to the (x_1, x_2) plane. Thus, equation (2-58) may be interpreted as the description of a phase plane oriented perpendicular to the (x_1, x_2) plane, or as a phase line characterizing the intersection of that phase plane with the (x_1, x_2) plane.

From the arguments presented above, it should be obvious that the two-dimensional wavevector, $\underline{k}$, is perpendicular to the phase line. Further, definitions and interpretations of the two-dimensional wavevector are easily obtained by specializing the relationships presented in section 2.1.1 to the case where k_3 is zero and x_3 is a constant.

The wavevector-frequency characterization of a wave field in one spatial dimension can also be developed as a special case of the three-dimensional spatial field. Consider the wave field, $p(\vec{x}, t)$, resulting from the wavevector-frequency transform

$$P(\vec{k},\omega) = (2\pi)^2 \tilde{P}(k_1,\omega)\delta(k_2)\delta(k_3) \ . \tag{2-59}$$

By equations (2-25) and (2-59), we obtain

$$p(\vec{x},t) = \tilde{p}(x_1,t) = (2\pi)^{-2} \iint_{-\infty}^{\infty} \tilde{P}(k_1,\omega)\exp[i(k_1x_1 + \omega t) \ dk_1 \ d\omega \ . \tag{2-60}$$

Equation (2-60) describes a space-time field, $\tilde{p}(x_1,t)$, that is a function of the single spatial variable, x_1, and time. By Fourier-transforming equation (2-60) in x_1 and t and by utilizing equation (2-38), one can easily show that

$$\tilde{P}(k_1,\omega) = \iint_{-\infty}^{\infty} \tilde{p}(x_1,t)\exp[-i(k_1x_1 + \omega t)] \ dx_1 \ dt \ . \tag{2-61}$$

Equations (2-60) and (2-61) constitute a Fourier transform pair.

As in the case of the wave field in two spatial dimensions, the wave field in one spatial dimension can be interpreted as a special case of the field in three spatial dimensions. By equation (2-59), all plane harmonic wave contributions to the space-time wave field are characterized by wavevectors having components $(k_1,0,0)$. As the wavevector has been shown to be directed perpendicular to the phase plane of the plane harmonic wave, it follows that the phase planes of all the harmonic wave contributions to $p(\vec{x},t)$ in equation (2-60) are perpendicular to the x_1 axis. Thus, the phases of the individual harmonic wave components in the integrand of equation (2-60) can be interpreted as descriptions of the kinematics of phase planes oriented perpendicular to the x_1 axis, or as the kinematics of the phase point defined by the intersection of the phase plane with the x_1 axis.

The definitions and kinematic interpretations of section 2.1.1 can be applied to the wave field in one spatial dimension by requiring k_2 and k_3 to be zero. These relations and definitions show that all harmonic components comprising the wave field in one spatial dimension are independent of x_2 and x_3 and propagate in the direction parallel to the x_1 axis and opposite to the direction of k_1.

2.4 REFERENCES

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CHAPTER 3
SPACE- AND TIME-INVARIANT LINEAR SYSTEMS

In the first chapter, wavevector-frequency analysis was defined as the description of space-time fields or systems in terms of their wavevector-frequency characteristics. The second chapter treated the description and interpretation of space-time fields in the wavevector-frequency domain. The characterization and interpretation of the response of systems in the wavevector-frequency domain will be the topic of the next five chapters.

The systems approach presented in these chapters parallels, in many aspects, the linear system theory developed by electrical engineers for the analysis of systems and fields that depend only on time. This approach was adopted because it provides a fundamental and consistent method of addressing a wide variety of problems, including those in acoustics.

This chapter reviews the basic concepts of systems theory and demonstrates the rationale for the wavevector-frequency analysis of one class of space-time systems: the space- and time-invariant system.

3.1 SYSTEMS AND THEIR CLASSIFICATIONS

The definition of a system that suits the purpose of this text is a combination of those found in The American Heritage Dictionary of the English Language[1] and Brogan's Modern Control Theory.[2] With suitable apologies to both sources, we define a system as an aggregation or assemblage of interacting elements combined by man or nature to form an integral entity.

The key word in this definition is "elements." If, for example, the elements are taken to be successive differential lengths of a string under tension, then some finite length of interest of the string can be considered to constitute a system. On the other hand, a finite length of string under tension would be an element of the system called the violin. Clearly, the

above definition of a system is sufficiently flexible to accommodate an infinite variety of components, interactions, and processes.

Systems theory is concerned with the interactions and behavior of the various elements of the system resulting from certain conditions or excitations imposed on the system. Therefore, the statement of a systems problem requires three definitions: (1) the definition of the elements and interactions comprising the system, (2) the definition of the conditions or inputs imposed on the system (usually called the input), and (3) the definition of the specific interaction or behavior of interest in the system, i.e., the system output. By this systems approach, a wide variety of problems can be reduced to the conceptually simple form depicted in figure 3-1.

In systems theory, a distinction is made between the physical system and the mathematical model of that system. The physical system is that assemblage of interacting devices, components, mechanisms, processes, etc., that have been selected for scrutiny. However, owing to cost considerations, the study of the behavior of the physical system under a given input is often conducted by means of an experimental or mathematical model of the system. Systems theory is concerned with the study and solution of these models of systems rather than the physical form of the system. Mathematical systems theory, or the study of mathematical models of systems, is the emphasis of these next few chapters.

The mathematical modeling of systems is an acquired skill, and a detailed discussion of the construction of mathematical models of systems is beyond the scope of this text. However, the mathematical form of the system model, including the forms of the input and consequent output, has a considerable impact on both the relative difficulty of predicting the output

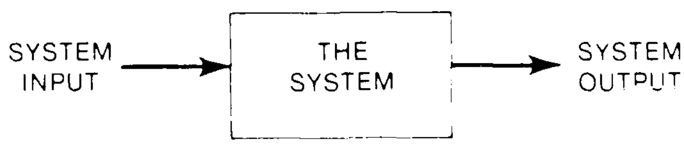

Figure 3-1. Conceptual Form of a Systems Problem

of the system for a given input and on the mathematical techniques required for that prediction. As a result, system models are usually classified according to the mathematical characteristics of the system model and its input. Table 3-1 lists the major characteristics used to classify mathematical models of space-time systems and their inputs.

Table 3-1. Major Classifications of Systems

	Mathematical Properties	
System Model	Linear	Nonlinear
	Deterministic	Stochastic
	Time Invariant	Time Varying
	Continuous Time	Discrete Time
	Space Invariant	Space Varying
	Spatially Distributed	Spatially Discrete
Input	Free	Forced
	Deterministic	Stochastic

In general, all of the factors in table 3-1 must be taken into account in the classification of the mathematical model of the system and its input. However, the order of the listing has no significance.

In this table, a linear system is one in which the equations governing all model elements are linear. If one or more elemental equations are nonlinear, the system is nonlinear.

Systems models that contain parameters which vary in some random fashion, and can be described only in terms of their statistical or average properties, are called stochastic systems. Otherwise, they are considered deterministic.

If the parameters of the mathematical model do not vary with time, the system is time invariant. If the mathematical model of the system is defined for all time, it is a continuous-time model. On the other hand, a model that treats the system only at discrete time intervals is a discrete-time model.

A space-invariant system is one in which the parameters of the mathematical model are independent of the spatial coordinates of the model. A mathematical model that describes the physical system continuously over space is a spatially distributed model, whereas one that treats the system only at discrete points in space is a spatially discrete model. Lumped parameter systems are a special case of spatially discrete systems.

Continuous space-time systems are modeled by partial differential equations, whereas discrete systems are modeled by ordinary differential equations or finite difference equations.

A system is said to be free if there are no external inputs to the system. In this case, the behavior of the system is completely determined by the system itself and its initial conditions. A forced system is one subject to external inputs. If either the external input or the initial conditions are subject to random variations, the input or initial conditions are considered stochastic.

3.2 CLASSIFICATION OF ACOUSTIC SYSTEMS

The systems of interest in this book are those associated with linear acoustics. Further, because (1) the subject of this text is wavevector-frequency analysis and (2) the wavevector and frequency are the respective Fourier conjugates of the spatial vector and time variables, our focus is on those acoustic systems that are continuous in time and spatially distributed.

Our purpose of teaching the fundamentals of wavevector-frequency analysis is best served by restricting attention to the deterministic, time-invariant mathematical models that describe the bulk of acoustic systems. Therefore, we will not attempt to address the acoustics of stochastic and time-varying media.

Infinite, spatially invariant models of systems are often employed in acoustics because they are relatively easy to solve and offer insight regarding the relative importance of the various physical processes influencing the system output. However, in many acoustics problems, the

effects of the spatial limitations and boundary conditions on the system output are the focus of the modeling effort. Such space-bounded models fall in the class of space-varying systems. Clearly, both space-varying and space-invariant acoustic systems must be addressed.

Although stochastic systems will not be treated in this text, considerable interest and history exists in the response of acoustic systems to stochastic inputs. Therefore, all forms of inputs will be considered.

In summary, the acoustic systems treated here will be limited to those that are linear, deterministic, time-invariant and continuous, and spatially distributed. However, all classifications of inputs will be treated.

This chapter treats the response of space- and time-invariant linear systems to deterministic inputs and initial conditions. Chapter 4 addresses the response of space-varying (but time-invariant) systems to deterministic inputs and initial conditions. Chapter 5 reviews some coupled systems of interest in structural acoustics. Chapter 6 develops the statistical concepts and descriptors required for the treatment of the response of systems to stochastic inputs or initial conditions, and chapter 7 deals with the response of systems to such random excitation.

3.3 FREE RESPONSE OF SPACE- AND TIME-INVARIANT LINEAR SYSTEMS

Spatially distributed, continuous-time systems in linear acoustics are modeled by linear partial differential equations in which the independent variables are spatial coordinates and time. If attention is further restricted to systems that are invariant in space and time, the coefficients of the various terms of these linear partial differential equations are constants.

In free systems, the absence of external inputs is reflected in the partial differential equations that model the system by the absence of inhomogeneous terms. Thus, the mathematical models of free space-time invariant linear systems are homogeneous partial differential equations with constant coefficients.

The output of a free system exists for all time and is sustained by natural interactions within the system. In the absence of external inputs, the initiation of the free response cannot be addressed. However, by specific knowledge of the output at any given time, the output can be determined for all time.

The outputs of free systems with losses cannot be described in the wavevector-frequency domain. The amplitudes of such outputs decrease monotonically with increasing time, and Fourier transforms of such outputs, over all time, do not exist.

The outputs of lossless, free space-time-invariant linear systems, however, can be described equivalently in the space-time domain or the wavevector-frequency domain. This equivalence of description and the techniques for solution in the wavevector-frequency domain can be demonstrated by some illustrative examples.

3.3.1 The Infinite String

A classical problem in linear acoustics is the free vibration of a uniform, infinitely long string, resulting from some specified initial displacement and velocity distribution. Here, the system is the string, free from external input, and the desired information (i.e., the output) is the space-time displacement field, $w(x,t)$, of the string. The mass per unit length (ε) and the tension (T) of the string are constant over the length of the string. The mathematical model describing the displacement of the string is given[3] by the following linear partial differential equation with constant coefficients:

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad (3\text{-}1)$$

for all x and t, where $c_s^2 = T/\varepsilon$.

If $w(x,t)$ is written as the wavenumber frequency transform

$$w(x,t) = (2\pi)^{-2} \iint_{-\infty}^{\infty} W(k,\omega) \exp\{i(kx + \omega t)\}\, dk\, d\omega \ , \qquad (3\text{-}2)$$

then it follows from equation (3-1) that $W(k,\omega)$ must satisfy

$$[(\omega/c_s)^2 - k^2]W(k,\omega) = 0 \qquad (3\text{-}3)$$

for all k and ω. Equation (3-3) states that $W(k,\omega)$ can have a nonzero solution only along the two lines defined by $|k| = |\omega/c_s|$, which are depicted in figure 3-2. Because of this restriction on the wavenumber content

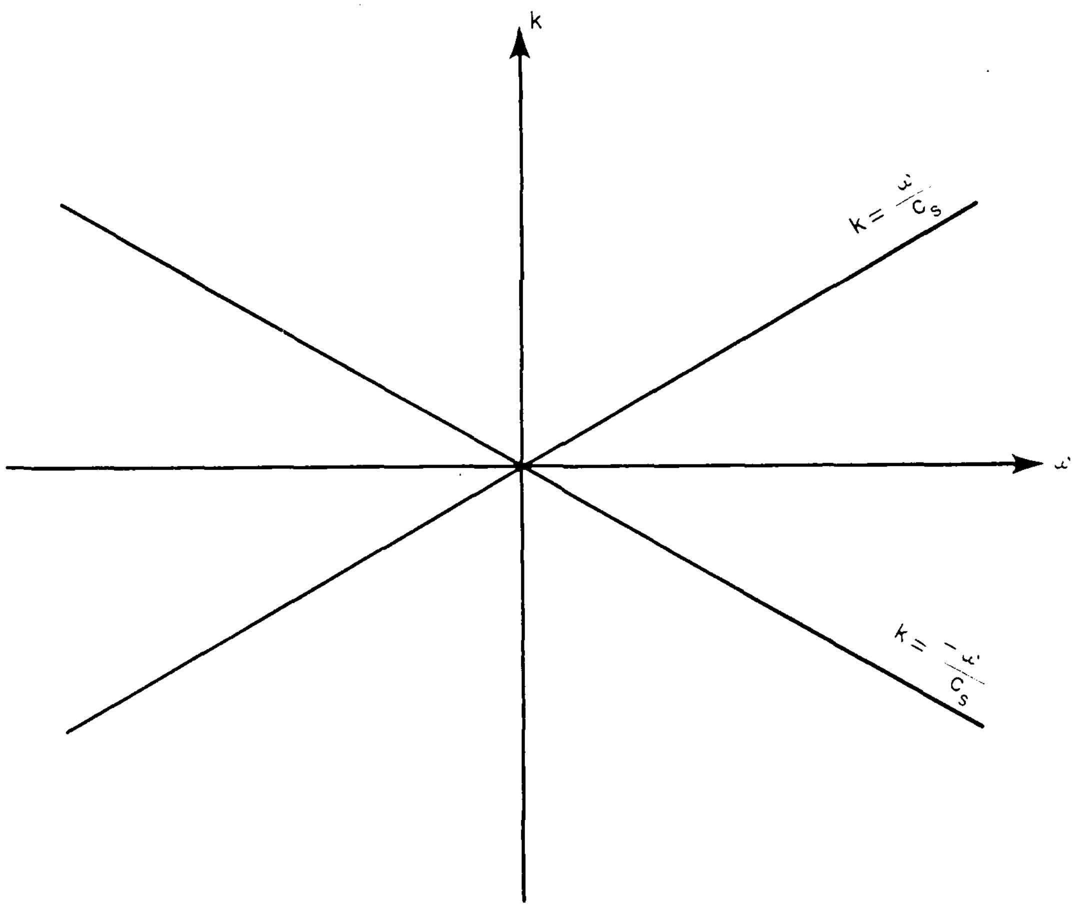

Figure 3-2. Locus of Wavevectors Characterizing Free Waves of an Infinite String as a Function of Frequency

of $W(k,\omega)$ at any frequency, the wavenumber

$$k_s = |\omega/c_s| \tag{3-4}$$

is called the free wavenumber of the string.

It is apparent, by equations (3-2) and (3-3), that if a particular wavenumber component, say k', is present in $W(k,\omega)$, its contributions to $w(x,t)$ can only be complex harmonic waves of the forms $\exp\{ik'(x + c_s t)\}$ and $\exp\{ik'(x - c_s t)\}$. The amplitudes of these harmonic wave components depend, of course, on the exact form of $W(k,\omega)$.

The mathematical form of $W(k,\omega)$ can be deduced by first writing $w(x,t)$ in the form

$$w(x,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \widetilde{W}(k,t)\exp(ikx)\, dk \ . \tag{3-5}$$

Substitution of equation (3-5) into equation (3-1) yields the ordinary differential equation

$$\frac{d^2\widetilde{W}(k,t)}{dt^2} + (kc_s)^2\, \widetilde{W}(k,t) = 0 \ , \tag{3-6}$$

which has the solution

$$\widetilde{W}(k,t) = A(k)\exp(ikc_s t) + B(k)\exp(-ikc_s t) \ . \tag{3-7}$$

By performing the temporal Fourier transformation of equation (3-7), we obtain

$$W(k,\omega) = 2\pi\{A(k)\delta(\omega - kc_s) + B(k)\delta(\omega + kc_s)\} \ . \tag{3-8}$$

Note that this solution to equation (3-3) is consistent, in form, with the solution to equation (2-50) given by equation (2-52). In equation (2-50), ω_0 was tacitly assumed to be a constant and, consequently, A and B in equation (2-52) were constants. In equation (3-3), however, k is a variable.

Therefore, the quantities A and B modifying the delta functions in equation (3-8) must be functions of k.

Equation (3-8) is the form of the general solution for the vibration displacement of the free, infinite, uniform string in the wavenumber-frequency domain. Note that $W(k,\omega)$, the wavenumber-frequency transform of $w(x,t)$, is characterized in the k-ω plane by a weighted distribution of delta functions along the lines $|k| = k_s$. The particular weighting functions, $A(k)$ and $B(k)$, are determined by the initial distribution of displacement and velocity on the string. Before proceeding to the determination of $A(k)$ and $B(k)$ in terms of these initial conditions, it should be noted that, by substituting equation (3-8) into equation (3-2) and performing the required integrations, one obtains

$$w(x,t) = a(x + c_s t) + b(x - c_s t) \ , \tag{3-9}$$

where $a(x)$ and $b(x)$ denote the respective inverse Fourier transforms of $A(k)$ and $B(k)$. Equation (3-9) is the general form of the classical solution for the free vibration of the infinite, uniform string.[4]

Assume that the initial displacement and velocity of the string are given by

$$w(x,0) = w_0(x) \tag{3-10}$$

and

$$\frac{\partial w(x,0)}{\partial t} = v_0(x) \ . \tag{3-11}$$

By equation (3-2), equation (3-9), and the use of the inverse Fourier transform, it is easily shown that

$$A(k) = (1/2)\{W_0(k) + [1/(ikc_s)]V_0(k)\} \tag{3-12}$$

and

$$B(k) = (1/2)\{W_0(k) - [1/(ikc_s)]V_0(k)\} \ , \tag{3-13}$$

where $W_0(k)$ and $V_0(k)$ are the spatial Fourier transforms of $w_0(x)$ and $v_0(x)$, respectively. Thus, by equations (3-8), (3-12), and (3-13), we obtain

$$W(k,\omega) = \pi\{W_0(k)[\delta(\omega - kc_s) + \delta(\omega + kc_s)] + [1/(ikc_s)]V_0(k)[\delta(\omega - kc_s) - \delta(\omega + kc_s)]\} . \qquad (3\text{-}14)$$

Equation (3-14) is the wavenumber-frequency description of the displacement field resulting from the free vibration of an infinite, uniform string with arbitrary initial displacement and velocity conditions. Recall, from section 2.2.1, that this wavenumber-frequency transform defines the amplitudes and initial phases of the harmonic waves comprising the space-time field as a function of the wavenumber and frequency characterizing each wave.

As noted previously, the wavenumber-frequency contributions to the displacement field consist of a weighted distribution of delta functions along the lines $|k| = k_s$. The weighting of the delta functions is completely determined by those wavenumber components comprising the initial displacement and velocity fields of the string.

In free vibration, only waves that result from natural interactions between elements of the string can be propagated. The delta functions of the form $\delta(\omega \pm kc_s)$ in each term of equation (3-14) are the mathematical statements of this restriction. These terms state that only waves characterized by wavenumbers and frequencies in the ratio $|\omega/k| = c_s$ (i.e., those with propagation speed c_s) can be propagated in the string. This restriction, implied by equation (3-3), is illustrated in figure 3-2.

As is evident by both equation (3-14) and figure 3-2, only two frequencies, equal in magnitude and opposite in sign, are associated with each wavenumber component of $W(k,\omega)$. By equation (2-14), this implies that each wavenumber component associated with the initial displacement and velocity fields contributes two harmonic waves to the vibration displacement field of the string: one propagating in the positive x direction and one propagating in the negative x direction. The speeds of propagation of both waves are easily shown (by equations (2-14) and (3-4)) to be independent of both

wavenumber and frequency and to be equal to c_s. The amplitudes and initial phases of these two waves are determined by their respective complex amplitudes, $W_0(k) - V_0(k)/(ikc_s)$ and $W_0(k) + V_0(k)/(ikc_s)$. The space-time displacement field of the vibrating string is the superposition of all such wave pairs dictated by the wavenumber content of the initial displacement and velocity fields.

The space-time field is obtained by substituting equation (3-14) into equation (3-2) and performing the integration on ω. This yields

$$w(x,t) = (4\pi)^{-1} \int_{-\infty}^{\infty} \{[W_0(k) + V_0(k)/(ikc_s)]\exp[ik(x + c_s t)] + [W_0(k) - V_0(k)/(ikc_s)]\exp[ik(x - c_s t)]\} \, dk \, . \tag{3-15}$$

It is easily demonstrated that

$$\frac{\exp(iku)}{ik} = \int_0^u \exp(iky) \, dy + \frac{1}{ik} \, . \tag{3-16}$$

By substitution of the appropriate form of this result in equation (3-15), the integrals over k are immediately recognized as simple inverse Fourier transforms of $W_0(k)$ and $V_0(k)$. It may thereby be shown that equation (3-15) reduces to the form

$$w(x,t) = (1/2)\left\{ w_0(x - c_s t) + w_0(x + c_s t) - \frac{1}{c_s} \int_0^{x-c_s t} v_0(y) \, dy + \frac{1}{c_s} \int_0^{x+c_s t} v_0(y) \, dy \right\} . \tag{3-17}$$

Equation (3-17) is the solution to the vibration of the infinite, uniform string obtained by traditional methods and presented in Morse.[4] That this

space-time solution was obtained by appropriate integration of the wavenumber-frequency description of the vibration field reinforces the assertion that both the space-time and wavenumber-frequency descriptions of a field contain equivalent information.

As a final observation, it should be noted that the solution for the free vibration of the string given by equation (3-17) is valid for all time. The explicit absence of external forces in the free system model precludes any consideration of how the vibratory motion was initiated. The initial conditions are therefore only simultaneous "snapshots" in time of the displacement and velocity fields. However, in the absence of external inputs, these initial conditions provide sufficient information to determine the vibration field prior to, as well as after, the time of the snapshot.

3.3.2 The Infinite Flat Plate

The technique for obtaining the wavenumber-frequency or space-time solution for the free response of space- and time-invariant systems is independent of the number of independent spatial variables required to mathematically model the system. To demonstrate this assertion and to introduce the concept of dispersive waves, we next treat the free transverse vibrations of an infinite, uniform, thin, flat plate.

The space-time field of the displacement of the central plane of the plate, measured normal to that plane, is designated by $w(\underline{x},t)$, where $\underline{x} = [x_1, x_2]$. The free vibration of the thin plate is governed by[5]

$$D\nabla^4 w + \mu \frac{\partial^2 w}{\partial t^2} = 0 , \qquad (3\text{-}18)$$

where

$$\nabla^4 = \left\{ \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}} \right\}^2 \qquad (3\text{-}19)$$

and where, for this spatially invariant system, the flexural rigidity (D) and the mass per unit area (μ) of the plate are constants.

If one assumes that the displacement field can be written in the form

$$w(\underline{x},t) = (2\pi)^{-3} \iint_{-\infty}^{\infty} W(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega\ , \tag{3-20}$$

substitution of equation (3-20) into equation (3-18) yields (as the resulting integral must hold for all $\underline{x}$ and t)

$$(k^4 - \frac{\mu}{D}\,\omega^2)W(\underline{k},\omega) = 0\ , \tag{3-21}$$

where $k^4 = (k_1^{\,2} + k_2^{\,2})^2$.

Equation (3-21) states that $W(\underline{k},\omega)$ must be zero except at those wavevectors having magnitudes equal to the fourth root of $\mu\omega^2/D$. Therefore, as we found in the case of the free vibration of the infinite string, only those waves that result from natural interactions between the elements of the infinite plate can contribute to its free vibration. By equation (3-21), only waves associated with wavevectors of specific magnitudes can contribute to the motion of the plate at each frequency. This wavevector magnitude, called the free wavenumber of the plate, is designated by k_p and given by

$$k_p = \sqrt[4]{\mu\omega^2/D}\ . \tag{3-22}$$

From equation (3-22) and the above discussion, it is apparent that the locus of all wavevectors contributing to the vibration of the plate at any given frequency must fall on a circle of radius k_p in the $\underline{k}$ plane and that the radius of that circle increases according to the square root of the magnitude of the frequency. However, according to equation (2-14), this implies that the phase speed (i.e., the magnitude of the phase velocity) of the waves comprising the free motion of the plate is a function of frequency and is given by

$$c_p = \sqrt[4]{D\omega^2/\mu}\ . \tag{3-23}$$

The quantity c_p, which is referred to as the free wave speed of the plate, is seen to increase with the square root of the magnitude of the frequency. This is in contrast to the free waves in the infinite string, which had a constant phase speed. Waves characterized by a phase speed that varies with frequency are called dispersive waves.

By equations (3-4) and (3-22), it is evident that the dispersive nature of a wave is reflected in the frequency dependence of the free wavenumber. That is, for the nondispersive waves in the uniform string, the free wavenumber is linearly related to the frequency, as indicated by equation (3-4). For the dispersive waves of the flat plate, the free wavenumber varies nonlinearly with frequency, as evidenced by equation (3-22).

The difference in the wavenumber-frequency characteristics of free waves in dispersive and nondispersive systems is illustrated in figure 3-3. Here, the free wavenumbers of the string (k_s) and the flat plate (k_p) are shown as a function of frequency. The nonlinear behavior of the (dispersive) free wavenumber of the plate with frequency is easily seen in contrast with the linear behavior associated with the (nondispersive) free wavenumber of the string.

Returning now to equation (3-21), the mathematical form of $W(\underline{k},\omega)$ can be determined by an extension of equations (2-50) and (2-51) or by assuming a form for $w(\underline{x},t)$ similar to that assumed for the displacement field of the string in equation (3-5). That is, in the latter approach, we assume

$$w(\underline{x},t) = (2\pi)^{-2} \int_{-\infty}^{\infty} \tilde{W}(\underline{k},t)\exp\{i\underline{k}\cdot\underline{x}\}\, d\underline{k} \ . \qquad (3\text{-}24)$$

Substitution of equation (3-24) into equation (3-18) yields the ordinary differential equation

$$\frac{d^2\tilde{W}(\underline{k},t)}{dt^2} + \frac{Dk^4}{\mu}\tilde{W}(\underline{k},t) = 0 \ , \qquad (3\text{-}25)$$

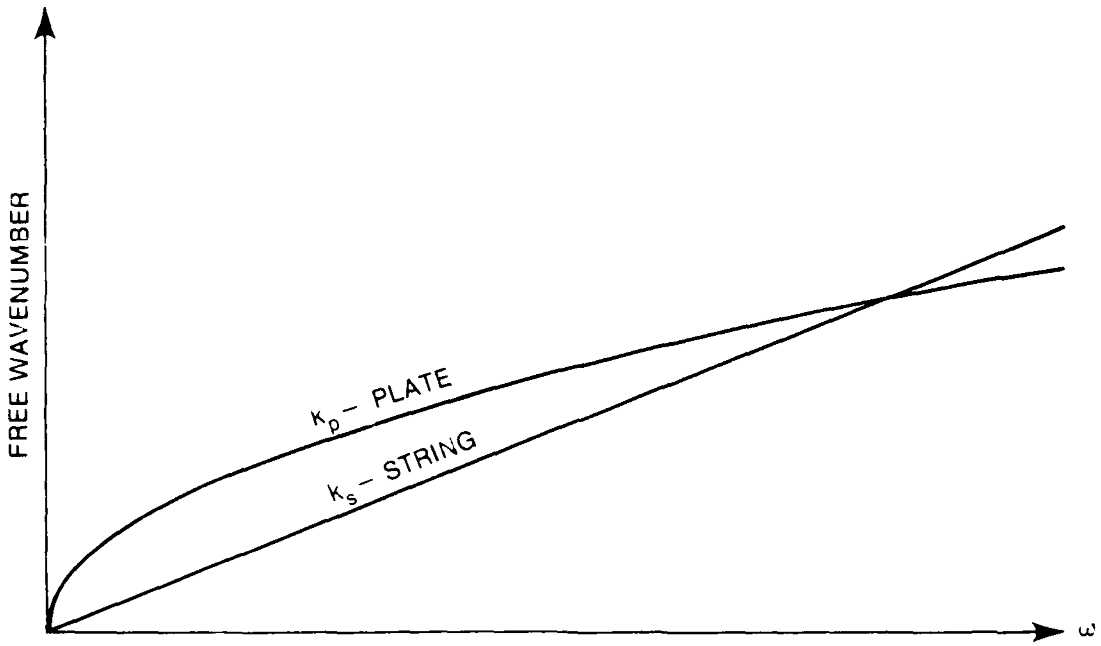

Figure 3-3. Comparison of the Free Wavenumbers of an Infinite Flat Plate and an Infinite String

which has the general solution

$$\tilde{W}(k,t) = A(\underline{k})\exp\{i\sqrt{D/\mu}\,k^2t\} + B(\underline{k})\exp\{-i\sqrt{D/\mu}\,k^2t\}\ , \qquad (3\text{-}26)$$

where A and B are arbitrary functions of $\underline{k}$ and $k^2 = k_1^2 + k_2^2$. By taking the temporal Fourier transform of equation (3-26), we obtain

$$W(\underline{k},\omega) = 2\pi[A(\underline{k})\delta\{\omega - \sqrt{D/\mu}\,k^2\} + B(\underline{k})\delta\{\omega + \sqrt{D/\mu}\,k^2\}]\ , \qquad (3\text{-}27)$$

and by equation (3-20),

$$w(\underline{x},t) = (2\pi)^{-2}\int_{-\infty}^{\infty}\Big[A(\underline{k})\exp\{i[\underline{k}\cdot\underline{x} + \sqrt{D/\mu}\,k^2t]\}$$

$$+ B(\underline{k})\exp\{i[\underline{k}\cdot\underline{x} - \sqrt{D/\mu}\,k^2t]\}\Big]\,d\underline{k}\ . \qquad (3\text{-}28)$$

As was the case in the vibrating string, the functions $A(\underline{k})$ and $B(\underline{k})$ are determined by the initial conditions of the plate. If the initial

displacement and velocity fields of the plate are given by

$$w(\underline{x},0) = w_0(\underline{x}) \tag{3-29}$$

and

$$\frac{dw(\underline{x},0)}{dt} = v_0(\underline{x}) \ , \tag{3-30}$$

then, by equation (3-28) and the definition of the Fourier transform, it is easily shown that

$$W(\underline{k},\omega) = \pi W_0(\underline{k})\{\delta[\omega - \sqrt{D/\mu}\ k^2] + \delta[\omega + \sqrt{D/\mu}\ k^2]\}$$

$$+ \frac{\pi V_0(\underline{k})}{i\sqrt{D/\mu}\ k^2}\{\delta[\omega - \sqrt{D/\mu}\ k^2] - \delta[\omega + \sqrt{D/\mu}\ k^2]\} \ , \tag{3-31}$$

where $W_0(\underline{k})$ and $V_0(\underline{k})$ are the respective spatial Fourier transforms of $w_0(\underline{x})$ and $v_0(\underline{x})$. Equation (3-31) is the wavevector-frequency description of the displacement field resulting from the free vibration of the infinite plate.

The physical interpretation of equation (3-31) parallels that of the somewhat similar mathematical form obtained, in equation (3-14), for the wavenumber-frequency description of the free vibration of the infinite string. That is, we first recall, from chapter 2, that $W(\underline{k},\omega)$ defines the amplitudes and initial phases of all the harmonic waves comprising the vibration field as a function of the wavevectors and frequencies characterizing each wave.

By equation (3-31), it is apparent that the wavevector content of the displacement field is completely determined by those wavevectors comprising the initial displacement and velocity fields: that is, those defined by $W_0(\underline{k})$ and $V_0(\underline{k})$. Further, only two frequencies, equal in magnitude and opposite in sign, are associated with each wavevector component of the initial displacement or velocity fields. The magnitudes of these frequencies are proportional to the squared magnitude of the wavevector. Thus, by equations

(2-14) and (3-20), each wavevector component of the initial displacement or velocity contributes two harmonic waves to the space-time displacement field of the plate; one wave propagates in the direction of the wavevector associated with that component and the other propagates opposite to that direction. The speeds of propagation of both components are equal and, by equations (3-22) and (3-23), proportional to the magnitude of the wavevector characterizing the component. The amplitudes and initial phases of these two waves are specified, respectively, by $W_0(\underline{k}) \pm iV_0(\underline{k})/[\sqrt{D/\mu}\, k^2]$. The space-time displacement field associated with the free vibration of the infinite plate is the superposition of all such wave pairs dictated by the wavevector content of the initial displacement and velocity fields.

As an illustrative example, consider the free vibration of the infinite thin plate resulting from the initial conditions

$$w_0(\underline{x}) = \alpha \sin(k_{10} x_1)$$

and (3-32)

$$v_0(\underline{x}) = 0 \ ,$$

where α is the (real) amplitude of the initial displacement and k_{10} is a constant wavenumber in the k_1 direction. It follows, by taking the spatial Fourier transforms of $w_0(\underline{x})$ and $v_0(\underline{x})$, that

$$W_0(\underline{k}) = (2\pi^2\alpha/i)\delta(k_2)\{\delta(k_1 - k_{10}) - \delta(k_1 + k_{10})\}$$

and (3-33)

$$V_0(\underline{k}) = 0 \ .$$

Therefore, only two wavevector components, with amplitudes equal in magnitude and opposite in phase, are present in the initial displacement and velocity. By substituting equation (3-33) into equation (3-31), we obtain

$$W(\underline{k},\omega) = (2\pi^3\alpha/i)\delta(k_2)\{\delta(k_1 - k_{10}) - \delta(k_1 + k_{10})\}$$
$$\{\delta[\omega - \sqrt{D/\mu}\, k_{10}^2] + \delta[\omega + \sqrt{D/\mu}\, k_{10}^2]\} \ . \quad (3\text{-}34)$$

By equation (3-34) and the previous discussion, we see that the free vibration field of the plate is comprised of the sum of four complex harmonic waves. The magnitudes of the (complex) amplitudes of all waves are equal, as are the magnitudes of the wavevectors and frequencies characterizing these waves. By use of equation (2-14), it can be shown that two of these waves propagate in the positive x_1 coordinate direction and the other two propagate in the negative x_1 direction. The speeds of propagation of all waves are equal and can be shown to be $\sqrt{D/\mu}\,|k_{10}|$. Finally, it is easily demonstrated that the wavevector-frequency description of this field has conjugate symmetry in the $\underline{k} - \omega$ domain; that is, $W(-\underline{k},-\omega) = W^*(\underline{k},\omega)$. Therefore, by the arguments of section 2.2.1, the space-time displacement field associated with this example of the free vibration of an infinite plate is real.

The space-time displacement field for this example is easily shown to be

$$w(x,t) = (\alpha/2)\{\sin[k_{10}x_1 + \sqrt{D/\mu}\,k_{10}^2 t] + \sin[k_{10}x_1 - \sqrt{D/\mu}\,k_{10}^2 t]\} . \tag{3-35}$$

Equation (3-35) shows that the space-time displacement field is the result of the spatial waveform of the initial displacement field being propagated, at half the initial amplitude, in both the positive and negative x_1 coordinate directions. The speed of propagation in both directions is equal and is that identified above.

3.3.3 Summary of Free Response Characteristics

The free response of the systems described in the above examples exhibits certain wavevector-frequency characteristics that are common to all free space- and time-invariant linear acoustic systems. In this section, we briefly summarize those characteristics.

The wavevector-frequency response of free space- and time-invariant systems defines the specific combination of free waves that comprise the space-time field associated with the system output. Free waves are that restricted set of waves which propagate in a system as a result of only the

natural reactions within the system. Inasmuch as (1) each wavevector-frequency combination defines a specific wave and (2) free waves are a restricted set, it follows that the set of wavevectors and frequencies that can contribute to the free response of a system is a restricted set.

Space- and time-invariant systems are infinite in spatial extent and have uniform properties in both space and time. It is therefore illogical, in the absence of external constraints or conditions, that there should be any preferred direction of propagation of free waves in the system. Recall, by the arguments of chapter 2, that the direction of propagation is determined by the direction of the wavevector and the sign of the frequency. It therefore follows that the wavevectors and frequencies that characterize free waves can only be restricted in their magnitudes.

A mathematical definition of the wavevectors and frequencies that characterize the free waves of a system can always be obtained by a multiple Fourier transformation, in all independent variables, of the partial differential equations governing the response, or output, of the system. The resulting equation relates the magnitudes of the wavevectors and frequencies that constitute free waves. The free wavenumber is defined as the magnitude of those wavevectors that constitute free waves at any particular frequency.

In the absence of external forces or inputs, the only wavevectors that can contribute to the output of the system are those present in the initial conditions. The initial conditions define the complex amplitude of each wavevector component that contributes to the free response of the system at some specified initial time. By knowledge of the wavevector components present in the initial conditions and the combinations of frequencies and wavevectors comprising free waves (by the definition of the free wavenumber), the wavevector-frequency content of the system output can be determined.

The illustrative examples presented above demonstrate a consistent mathematical procedure for obtaining the wavevector-frequency description of the free response of the space- and time-invariant systems encountered in linear acoustics.

3.4 FORCED RESPONSE OF SPACE- AND TIME-INVARIANT LINEAR SYSTEMS

The mathematical models of forced space- and time-invariant linear systems differ from those of their free counterparts only by the addition of the forcing term, or input, that is not a function of the independent, or output, variable. Thus, by the arguments of section 3.3, the mathematical models of these forced systems are inhomogeneous linear partial differential equations with constant coefficients.

This section describes a fundamental and consistent technique for obtaining and interpreting the wavevector-frequency response of forced space- and time-invariant linear acoustic systems.

3.4.1 The Principle of Superposition in Linear Systems

The solution for the forced response of linear systems is based on the principle of superposition for linear equations. Let $L_{\vec{x},t}$ denote any linear partial differential operator of the form

$$L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J} \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{n=0}^{N} a_{jlmn}(\vec{x},t) \left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} , \tag{3-36}$$

where $\vec{x} = (x_1,x_2,x_3)$ and the indices j, l, m, and n denote the order of the partial derivatives in x_1, x_2, x_3, and t, respectively. Mathematical descriptions of linear systems in acoustics are characterized by operators of the form of equation (3-36).

If $p(\vec{x},t)$ denotes the output of a linear system resulting from the input $f(\vec{x},t)$, the inhomogeneous linear partial differential equation that describes this input output relationship is given by

$$L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t) . \tag{3-37}$$

If $p_1(\vec{x},t)$, $p_2(\vec{x},t)$, ..., $p_N(\vec{x},t)$ are the solutions of equation (3-37)

resulting from the separate inputs $f_1(\vec{x},t)$, $f_2(\vec{x},t)$, ..., $f_N(\vec{x},t)$, then it follows that

$$\sum_{n=1}^{N} b_n L_{\vec{x},t}\{p_n(\vec{x},t)\} = \sum_{n=1}^{N} b_n f_n(\vec{x},t) \ , \tag{3-38}$$

where the constants b_n are arbitrary. However, owing to the form of $L_{\vec{x},t}$ shown in equation (3-36), it is easily seen that equation (3-38) can be rewritten in the equivalent form

$$L_{\vec{x},t}\left\{\sum_{n=1}^{N} b_n p_n(\vec{x},t)\right\} = \sum_{n=1}^{N} b_n f_n(\vec{x},t) \ . \tag{3-39}$$

By equations (3-37) and (3-39), it is clear that if the input to a system is a linear combination of the form

$$f(\vec{x},t) = \sum_{n=1}^{N} b_n f_n(\vec{x},t) \ , \tag{3-40}$$

then the output is given by

$$p(\vec{x},t) = \sum_{n=1}^{N} b_n p_n(\vec{x},t) \ . \tag{3-41}$$

Equations (3-40) and (3-41) are a mathematical statement of the principle of superposition for linear systems, which forms the foundation for the treatment of forced linear systems.

For the space- and time-invariant systems of interest in this chapter, the partial differential equations governing the system have constant coefficients. Thus, for space- and time-invariant systems, the coefficients a_{jlmn} in the linear operator of equation (3-36) are constants. However, when a_{jlmn} are constants, it is easily seen that the form of the linear

operator of equation (3-36) is independent of the origins of the spatial and temporal coordinates. That is, if we define

$$\vec{\xi} = \vec{x} - \vec{x}_0$$

and (3-42)

$$\tau = t - t_0 ,$$

where $\vec{x}_0$ and t_0 are arbitrary constants, and if we denote the space- and time-invariant linear operator by ${}_I L_{\vec{x},t}$, it is easily shown that

$${}_I L_{\vec{\xi},\tau}\{\ \} = {}_I L_{\vec{x},t}\{\ \} . \tag{3-43}$$

It follows, by equations (3-37) and (3-43), that

$${}_I L_{\vec{\xi},\tau}\{p(\vec{\xi} + \vec{x}_0,\tau + t_0)\} = f(\vec{\xi} + \vec{x}_0,\tau + t_0) , \tag{3-44}$$

from which it must be concluded that the output of a linear space- and time-invariant system resulting from the input $f(\vec{\xi} + \vec{x}_0,\tau + t_0)$ is $p(\vec{\xi} + \vec{x}_0,\tau + t_0)$.

By using these fundamental concepts of linear systems, a logical and consistent approach to obtaining solutions for the forced response of space- and time-invariant linear systems can be developed.

3.4.2 The Green's Function or Space-Time Impulse Response

The Green's function (also descriptively known as the space-time impulse response) of a system is defined as the response of that system at the spatial coordinate $\vec{x}$ and time t to an impulsive input applied at the spatial location $\vec{x}_0$ at time t_0. If we denote the Green's function by $g(\vec{x},t;\vec{x}_0,t_0)$ and assume that the system is governed by a linear inhomogeneous partial differential equation of the form of equation (3-37), it follows that the Green's function is mathematically defined by

$$L_{\vec{x},t}\{g(\vec{x},t;\vec{x}_0,t_0)\} = \delta(\vec{x} - \vec{x}_0)\delta(t - t_0) . \tag{3-45}$$

where

$$\delta(\vec{x} - \vec{x}_0) = \delta(x_1 - x_{01})\delta(x_2 - x_{02})\delta(x_3 - x_{03}) \ . \tag{3-46}$$

The argument of the Green's function in equation (3-45) is written in the traditional form and deserves some explanation. The independent variables of the Green's function, $\vec{x}$ and t, define the absolute spatial coordinates and time of observation of the output of the system. The parameters $\vec{x}_0$ and t_0 define the spatial coordinates and time of application of the impulsive input. Clearly, the mathematical form of the Green's function depends on both the observation variables and the input parameters. The inclusion of the input parameters in the argument of the Green's function serves as a reminder of this functional dependence.

In this chapter, our focus is only on space- and time-invariant linear systems. Therefore, by noting the form of the particular f in equation (3-45), we can employ equation (3-44) to obtain

$${}_I L_{\vec{x},t}\{g(\vec{x} - \vec{x}_0, t - t_0)\} = \delta(\vec{x} - \vec{x}_0)\delta(t - t_0) \ . \tag{3-47}$$

By equation (3-47), it is clear that, for space- and time-invariant linear systems, the Green's function has the mathematical form

$$g(\vec{x},t;\vec{x}_0,t_0) = g(\vec{x} - \vec{x}_0, t - t_0) \tag{3-48}$$

and thereby depends only on the difference between the variables of observation and the parameters of excitation.

By use of the sampling property of the Dirac delta function (see equation (2-31)), we may express any system input, $f(\vec{x},t)$, as

$$f(\vec{x},t) = \iint_{-\infty}^{\infty} f(\vec{x}_0,t_0)\delta(\vec{x} - \vec{x}_0)\delta(t - t_0)\ d\vec{x}_0\ dt_0 \ , \tag{3-49}$$

where $d\vec{x}_0$ denotes $dx_{01}dx_{02}dx_{03}$. It is easily shown, from equations (3-39), (3-47), and (3-49), that the response of a space- and time-invariant linear

system to an arbitrary input, $f(\vec{x},t)$, is governed by

$$I L_{\vec{x},t}\left\{\iint_{-\infty}^{\infty} f(\vec{x}_0,t_0)g(\vec{x}-\vec{x}_0,t-t_0)\,d\vec{x}_0\,dt_0\right\}$$

$$= \iint_{-\infty}^{\infty} f(\vec{x}_0,t_0)\delta(\vec{x}-\vec{x}_0)\delta(t-t_0)\,d\vec{x}_0\,dt_0 = f(\vec{x},t)\ . \qquad (3\text{-}50)$$

It therefore follows, by the definition of equation (3-39), that the output, $p(\vec{x},t)$, of a space- and time-invariant linear system to any input, $f(\vec{x},t)$, is given by

$$p(\vec{x},t) = \iint_{-\infty}^{\infty} f(\vec{x}_0,t_0)g(\vec{x}-\vec{x}_0,t-t_0)\,d\vec{x}_0\,dt_0\ . \qquad (3\text{-}51)$$

By employing the change of variables of equation (3-42), we may write equation (3-51) in the equivalent form

$$p(\vec{x},t) = \iint_{-\infty}^{\infty} f(\vec{x}-\vec{\xi},t-\tau)g(\vec{\xi},\tau)\,d\vec{\xi}\,d\tau\ . \qquad (3\text{-}52)$$

Equations (3-51) and (3-52) show that, by knowledge of the Green's function of a space- and time-invariant linear system, the output of the system resulting from any input can, in principle, be obtained. The caveat "in principle" is stated because, in some cases, the integrals cannot be evaluated in closed form. However, these integral forms pose no problem for characterization of the output in the wavevector-frequency domain.

Up to this point, we have not addressed the question of the initial conditions used to uniquely define the Green's function. The linear acoustic systems treated in this book are causal systems. A causal system is one that is at rest until acted upon by an external input. Thus, the output of a causal system depends only on inputs that existed in past times; the system

does not respond in anticipation of future inputs. Therefore, for a causal system, it follows that

$$g(\vec{x},t;\vec{x}_0,t_0) = 0 \ , \quad t < t_0 \ ,$$

and (3-53)

$$\frac{\partial^n g(\vec{x},t;\vec{x}_0,t_0)}{\partial t^n} = 0 \ , \quad t < t_0 \text{ for all } n \ .$$

Equation (3-53) defines the initial conditions for the causal Green's function.

For causal space- and time-invariant linear systems, it follows from equation (3-53) that

$$g(\vec{x} - \vec{x}_0, t - t_0) = 0 \ , \quad t < t_0 \ ,$$

or (3-54)

$$g(\vec{\xi},\tau) = 0 \ , \quad \tau < 0 \ .$$

Therefore, for causal systems, the infinite upper limit of the temporal integral in equation (3-51) can be replaced by t, and the negative infinite lower limit on the temporal integral in equation (3-52) can be replaced by zero. Many texts and papers use these alternative limits in expressing system outputs in terms of Green's functions. However, in this text, we will continue to use infinite temporal limits and rely on those temporal properties of the causal Green's functions indicated by equation (3-54) to effectively limit the range of temporal integration.

It should be emphasized that, for the causal systems treated in this text, equations (3-51) and (3-52) describe the output of a system that is at rest (i.e., has zero output) until an external input is applied.

3.4.3 The Wavevector-Frequency Response

The wavevector-frequency transform of the output of the space- and time-invariant linear system can be related to the wavevector-frequency

transform of the input field by use of equation (3-52). That is, by writing

$$p(\vec{x},t) = (2\pi)^{-4} \iint_{-\infty}^{\infty} P(\vec{k},\omega)\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}\, d\vec{k}\, d\omega \tag{3-55}$$

and

$$f(\vec{x},t) = (2\pi)^{-4} \iint_{-\infty}^{\infty} F(\vec{k},\omega)\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}\, d\vec{k}\, d\omega , \tag{3-56}$$

we can rewrite equation (3-52) as

$$(2\pi)^{-4} \iint_{-\infty}^{\infty} \{P(\vec{k},\omega) - F(\vec{k},\omega)G(\vec{k},\omega)\}\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}\, d\vec{k}\, d\omega = 0 , \tag{3-57}$$

where

$$G(\vec{k},\omega) = \iint_{-\infty}^{\infty} g(\vec{\xi},\tau)\exp\{-i(\vec{k}\cdot\vec{\xi} + \omega\tau)\}\, d\vec{\xi}\, d\tau . \tag{3-58}$$

Inasmuch as equation (3-57) is valid for all space and time, it follows that

$$P(\vec{k},\omega) = F(\vec{k},\omega)G(\vec{k},\omega) . \tag{3-59}$$

This simple linear algebraic relation between the wavevector-frequency transforms of the input and output is in sharp contrast to the four-dimensional convolution required (in equation (3-52)) to specify the space-time output. Indeed, it is interesting to note, by inverse Fourier transformation of equation (3-59) and use of equation (3-52), that

$$(2\pi)^{-4} \iint_{-\infty}^{\infty} \{F(\vec{k},\omega)G(\vec{k},\omega)\}\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}\, d\vec{k}\, d\omega$$

$$= \iint_{-\infty}^{\infty} f(\vec{x} - \vec{\xi}, t - \tau)g(\vec{\xi},\tau)\, d\vec{\xi}\, d\tau . \tag{3-60}$$

This result is the four-dimensional extension of the convolution theorem expressed by equation (2-54).

The wavevector-frequency transform of the Green's function, $G(k,\omega)$, can be shown to have a simple physical interpretation. Consider the response of a space- and time-invariant linear system to the input

$$f(\vec{x},t) = \exp\{i(\vec{k}_0\cdot\vec{x} + \omega_0 t)\} \tag{3-61}$$

over all $\vec{x}$ and t, where $\vec{k}_0$ and ω_0 are constants. Substitution of equation (3-61) into equation (3-52) yields, by use of equation (3-58),

$$p(\vec{x},t) = G(\vec{k}_0,\omega_0)\exp\{i(\vec{k}_0\cdot\vec{x} + \omega_0 t)\} \ . \tag{3-62}$$

By equations (3-61) and (3-62), $G(\vec{k},\omega)$ is the ratio of the space-time output field of the system to the input field when the input is a complex harmonic plane wave of the form $\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}$. For that reason, $G(\vec{k},\omega)$ is called the wavevector-frequency response of the system.

Some texts on acoustics[6,7,8] employ the concept of mechanical and acoustic impedance. These impedances, based on a force-voltage analogy between acoustic and electrical systems, relate the force or pressure (as appropriate) input to a system to the consequent velocity output of the system under conditions of harmonic excitation. Many of the papers and reports dealing with the application of wavevector-frequency analysis use an impedance to relate the wavevector-frequency transform of the force or pressure to that of the velocity. Consider, for example, a space- and time-invariant linear system in which the input is a pressure field, $p(\vec{x},t)$, and the output is the velocity field, $v(\vec{x},t)$. The acoustic impedance is defined as

$$Z_a(\vec{k},\omega) = P(\vec{k},\omega)/V(\vec{k},\omega) \ , \tag{3-63}$$

where $P(\vec{k},\omega)$ and $V(\vec{k},\omega)$ are the respective wavevector-frequency transforms of $p(\vec{x},t)$ and $v(\vec{x},t)$. Similarly, if the system input is a force field, say $f(\vec{x},t)$, and the output is a velocity field, the mechanical impedance is

defined by

$$Z_m(\vec{k},\omega) = F(\vec{k},\omega)/V(\vec{k},\omega) \quad , \qquad (3\text{-}64)$$

where $F(\vec{k},\omega)$ is the wavevector transform of $f(\vec{x},t)$.

By comparing the forms of equations (3-63) and (3-64) to that of equation (3-59), it is obvious that the acoustic and mechanical impedances are simply the reciprocals of the wavevector-frequency response in these specialized applications. It follows then, by arguments similar to those employed in equations (3-61) and (3-62), that the acoustic and mechanical impedances are simply the ratio of the space-time pressure or force field, as appropriate, to the resultant velocity field when the pressure or force field is a single complex harmonic plane wave of the form $\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}$ for all $\vec{x}$ and t.

By the above arguments, the wavevector-frequency description (i.e., transform) of the output of a space- and time-invariant linear system is easily achieved, given the wavevector-frequency response of the system and the wavevector-frequency description of the forcing field. Alternatively, if one knows (by observation or measurement) the wavevector-frequency transform of the output and the wavevector-frequency response of the system, the wavevector-frequency characteristics of the input field can be deduced. Finally, by knowledge of the wavevector-frequency transforms of the input and output fields, the wavevector-frequency response of the system can be deduced.

To illustrate (1) the mathematical techniques for obtaining the wavevector-frequency response and (2) the interpretative advantages of the wavevector-frequency description of systems, we present the following illustrative examples of the forced response of space- and time-invariant linear systems.

3.4.4 The Forced Vibration of a Uniform Infinite String

Consider the displacement, w(x,t), of a uniform, infinitely long string resulting from a force per unit length, f(x,t), applied to the string. The mathematical model of this system is given by

$$T\,\frac{\partial^2 w}{\partial x^2} - \varepsilon\,\frac{\partial^2 w}{\partial t^2} = -f(x,t)\ , \tag{3-65}$$

where it will be recalled that T and ε are the respective (constant) tension and mass per unit length of the string. In equation (3-65), f(x,t) is considered positive when applied in the direction of positive w(x,t).

The Green's function for the uniform, infinite string is the solution to equation (3-65) when f(x,t) is replaced by $\delta(x - x_0)\delta(t - t_0)$. As equation (3-65) applies over all space and time, we define $\xi = x - x_0$ and $\tau = t - t_0$ and then write the equation for the Green's function, $g(\xi,\tau)$, as

$$\frac{\partial^2 g}{\partial \xi^2} - \frac{1}{c_s^2}\,\frac{\partial^2 g}{\partial \tau^2} = \frac{-\delta(\xi)\delta(\tau)}{\varepsilon c_s^2}\ , \tag{3-66}$$

where it will be recalled that $c_s^2 = T/\varepsilon$.

There are a variety of methods for obtaining the solution to equation (3-66). However, because our immediate goal is the determination of the wavenumber-frequency transform, $G(k,\omega)$, of the causal Green's function, we will use Fourier transform techniques to solve this equation.

To obtain the particular solution, denoted by g_p, to equation (3-66), we write

$$g_p(\xi,\tau) = (2\pi)^{-2} \iint_{-\infty}^{\infty} G_p(k,\omega)\exp\{i(k\xi + \omega\tau)\}\ dk\ d\omega\ . \tag{3-67}$$

Then, by using equations (2-38) and (3-67), equation (3-66) can be written in the form

$$(2\pi c_s)^{-2} \iint_{-\infty}^{\infty} \{[\omega^2 - (kc_s)^2]G_p(k,\omega) + 1/\varepsilon\}\exp\{i(k\xi + \omega\tau)\}\ dk\ d\omega = 0\ . \tag{3-68}$$

As equation (3-68) holds for all ξ and τ, it follows that the wavenumber-frequency transform of the particular solution is

$$G_p(k,\omega) = \frac{-1}{\epsilon[\omega^2 - (kc_s)^2]} . \qquad (3\text{-}69)$$

The wavenumber-frequency transform of the homogeneous solution to equation (3-66), denoted by $G_h(k,\omega)$, is precisely that developed for the free vibration of the string and given by equation (3-8). That is,

$$G_h(k,\omega) = 2\pi\{A(k)\delta(\omega - kc_s) + B(k)\delta(\omega + kc_s)\} . \qquad (3\text{-}70)$$

The wavenumber-frequency transform of the Green's function for the infinite string is the sum of the particular and homogeneous solutions, where the functions A(k) and B(k) are determined by the initial conditions.

The initial condition for the desired causal Green's function is $g(\xi,\tau) = 0$ for $\tau < 0$ for all ξ. If we define

$$\tilde{G}(k,\tau) = \int_{-\infty}^{\infty} g(\xi,\tau)\exp\{-ik\xi\}\, d\xi , \qquad (3\text{-}71)$$

it follows that the initial condition translates to $\tilde{G}(k,\tau) = 0$ for $\tau < 0$ for all k. From equations (3-69) and (3-70), $\tilde{G}(k,\tau)$ can be obtained by the inverse Fourier transformation

$$\tilde{G}(k,\tau) = (2\pi)^{-1} \int_{-\infty}^{\infty} \{G_p(k,\omega) + G_h(k,\omega)\}\exp\{i\omega\tau\}\, d\omega . \qquad (3\text{-}72)$$

Let us consider the inverse transforms of G_p and G_h separately.

By a partial fraction expansion of equation (3-69) and use of equation (3-72), the particular portion $\tilde{G}_p(k,\tau)$ of $\tilde{G}(k,\tau)$ may be written

$$\tilde{G}_p(k,\tau) = \frac{1}{4\pi\varepsilon k c_s} \int_{-\infty}^{\infty} \left\{\frac{1}{\omega + kc_s} - \frac{1}{\omega - kc_s}\right\} \exp\{i\omega\tau\}\, d\omega \ . \tag{3-73}$$

However,

$$\int_{-\infty}^{\infty} \frac{\exp\{i\omega\tau\}}{\omega + \beta}\, d\omega = \exp\{-i\beta\tau\} \int_{-\infty}^{\infty} \frac{\exp\{i\sigma\tau\}}{\sigma}\, d\sigma \ , \tag{3-74}$$

and Papoulis[9] shows that

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \frac{2\exp\{i\omega\tau\}}{i\omega}\, d\omega = \mathrm{sgn}(\tau) \ . \tag{3-75}$$

The generalized function $\mathrm{sgn}(\tau)$ in equation (3-75) is defined by

$$\mathrm{sgn}(\tau) = \begin{cases} 1 \ , & \tau > 0 \\ -1 \ , & \tau < 0 \end{cases} = 2U(\tau) - 1 \ , \tag{3-76}$$

where $U(\tau)$ is the Heaviside function defined in equation (2-32).

By equations (3-73), (3-74), and (3-75), it is straightforward to show that

$$\tilde{G}_p(k,\tau) = \frac{1}{2\varepsilon}\, \mathrm{sgn}(\tau)\, \frac{\sin(kc_s\tau)}{kc_s} \ . \tag{3-77}$$

Further, by equations (3-70) and (3-72), it can be shown that

$$\tilde{G}_h(k,\tau) = A(k)\exp\{ikc_s\tau\} + B(k)\exp\{-ikc_s\tau\} \ . \tag{3-78}$$

Inasmuch as $\tilde{G}(k,\tau) = \tilde{G}_p(k,\tau) + \tilde{G}_h(k,\tau)$ and causality requires that $\tilde{G}(k,\tau) = 0$ for $\tau < 0$ for all k, it follows, by equations (3-77) and (3-78), that

$$A(k) = -B(k) = 1/(4i\varepsilon k c_s) \ , \tag{3-79}$$

and thus, by use of equations (3-76), (3-77), (3-78), and (3-79),

$$\tilde{G}(k,\tau) = (1/\varepsilon)U(\tau)\,\frac{\sin(kc_s\tau)}{kc_s}\,. \qquad (3\text{-}80)$$

Equation (3-80) is the wavenumber transform of the causal Green's function for the infinite, uniform string.

Our interest is in the wavenumber-frequency rather than in the space-time description of the Green's function. However, for the sake of completeness, we note that the space-time description of the causal Green's function can be obtained by the inverse Fourier transformation of equation (3-80). By writing $\sin(kc_s\tau)$ in equation (3-80) in its exponential form and by using equations (3-75) and (3-76), one can show that

$$g(\xi,\tau) = [1/(4\varepsilon c_s)]U(\tau)\{\mathrm{sgn}(\xi + c_s\tau) - \mathrm{sgn}(\xi - c_s\tau)\} = [1/(2\varepsilon c_s)]U(c_s\tau - |\xi|)\,. \qquad (3\text{-}81)$$

Figure 3-4 depicts the Green's function for the infinite string as a function of ξ at a constant, but arbitrary, value of τ. The Green's function is usually interpreted as the output of the system resulting from an impulsive force. This is not strictly true inasmuch as equation (3-52) shows that the dimensions of the Green's function are not those of the output, or even the output divided by the input. However, if I is defined as a constant of magnitude one and dimensions of force-time, it can be argued (by equations (3-52) and (3-66)) that $Ig(\xi,\tau)$ is the displacement of the string resulting from the impulsive force per unit length $I\delta(x - x_0)\delta(t - t_0)$. Thus, it follows that $g(\xi,\tau)$ is proportional to the displacement resulting from the impulsive excitation.

Note, by figure 3-4, that at τ seconds after the applied impulse, $g(\xi,\tau)$ is constant at all spatial locations less than $|c_s\tau|$ from the point of application of the impulsive force and is zero at all spatial locations greater than $|c_s\tau|$. As time increases, the region of constant displacement increases, linearly with time, symmetrically about the point of excitation.

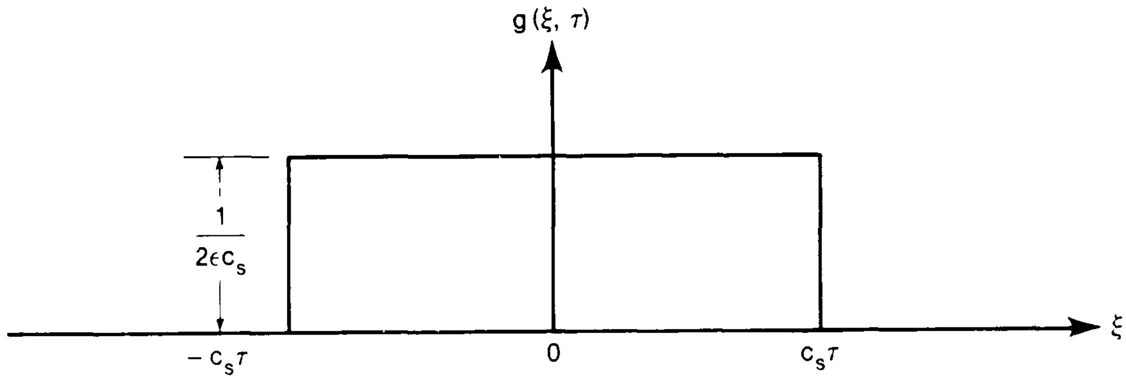

Figure 3-4. The Spatial Dependence of the Green's Function of an Infinite String for a Fixed τ

We now turn our attention to the wavenumber-frequency response of the infinite string. The wavenumber-frequency response, it will be recalled, is defined as the wavenumber-frequency transform of the causal Green's function. By equations (3-69), (3-70), and (3-79), the wavenumber-frequency response of the infinite, uniform string is given by

$$G(k,\omega) = \frac{-1}{\epsilon[\omega^2 - (kc_s)^2]} + \frac{\pi}{2i\epsilon kc_s}\{\delta(\omega - kc_s) - \delta(\omega + kc_s)\} . \qquad (3\text{-}82)$$

Figures 3-5(a) and (b) illustrate the real and imaginary parts, respectively, of $G(k,\omega)$ as a function of k at a fixed, but arbitrary, frequency, ω. Both real and imaginary parts of $G(k,\omega)$ are seen to be even functions of k. Note that the real part of $G(k,\omega)$ is the wavenumber-frequency transform of the particular part of the Green's function, and the imaginary part is the transform of the homogeneous part. Figure 3-5 shows that the wavenumber-frequency response is well-behaved, except at those wavenumbers where $|k| = |\omega/c_s|$. Let us therefore interpret the wavenumber-frequency response in this wavenumber range (i.e., $|k| \neq |\omega/c_s|$) first.

Recall, by equation (3-62), that the wavenumber-frequency response can be interpreted as the ratio of $w(x,t)$ to $f(x,t)$ when $f(x,t)$ is a single complex harmonic wave of the form $\exp\{i(kx + \omega t)\}$ for all x and t. Note, by equation (3-82) and figure 3-5, that when $|k| \neq |\omega/c_s|$, the imaginary part of $G(k,\omega)$ is zero. As the imaginary part corresponds to the homogeneous solution, the

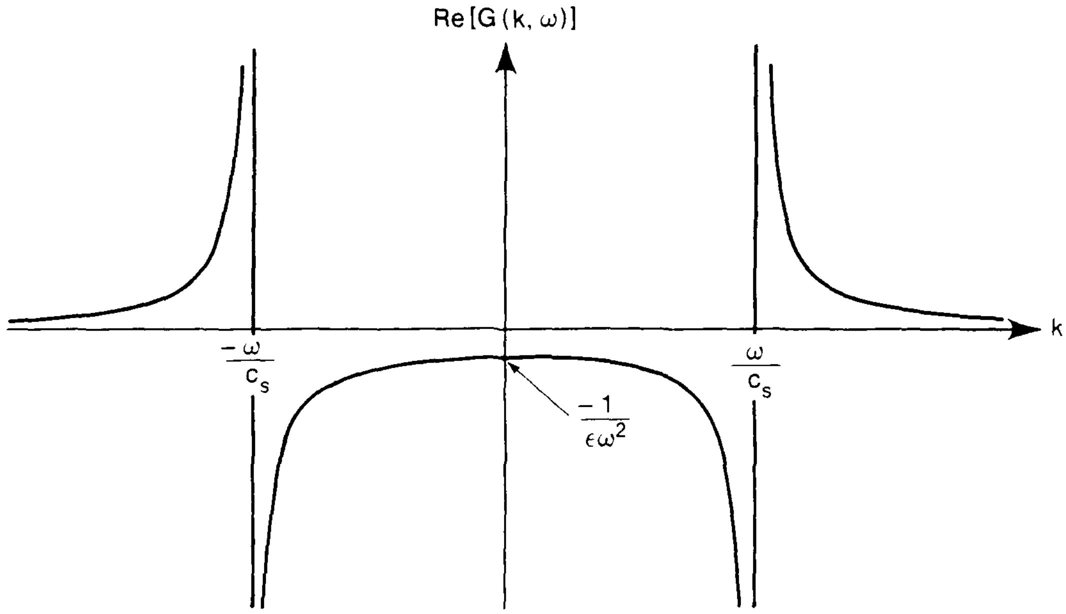

Figure 3-5(a). Real Part of Wavenumber-Frequency Response

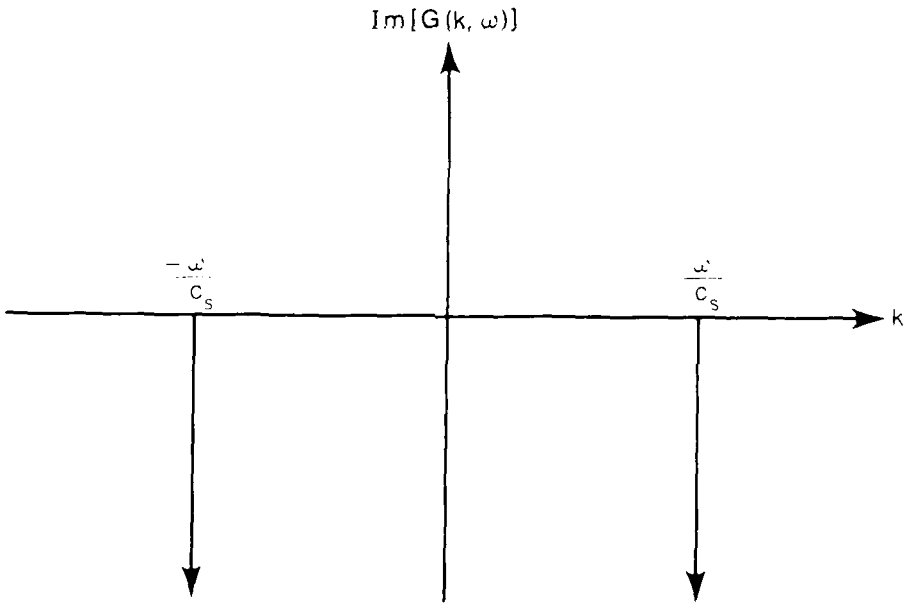

Figure 3-5(b). Imaginary Part of Wavenumber-Frequency Response

Figure 3-5. Real and Imaginary Parts of the Wavenumber-Frequency Response of an Infinite String

wavenumber-frequency response, over the range $|k| \neq |\omega/c_s|$, can be interpreted as the ratio of the particular solution of equation (3-65) to the forcing field $f(x,t) = \exp\{i(kx + \omega t)\}$. This particular solution corresponds to the real part of the wavenumber-frequency response illustrated in figure 3-5(a).

Consider now a fixed (but arbitrary) frequency of the harmonic wave excitation, $f(x,t) = \exp\{i(kx + \omega t)\}$. The particular solution to equation (3-65) is $w(x,t) = G_p(k,\omega)\exp\{i(kx + \omega t)\}$, where G_p, the amplitude of $w(x,t)$, is the real part of $G(k,\omega)$. By substituting this form of solution into equation (3-65), one can see that when the magnitude of the wavenumber of excitation is large compared with the free wavenumber (ω/c_s) of the string, the applied force is primarily balanced by the tensile forces in the string and the displacement, $w(x,t)$, is in phase with the applied force. For a wave in the string of the form $\exp\{i(k\xi + \omega\tau)\}$ and constant amplitude, tensile forces increase with increasing wavenumber magnitude (i.e., decreasing wavelength). Thus, in the wavenumber region $|k| > |\omega/c_s|$, where tensile forces dominate, the response of the string to the constant amplitude applied force must decrease with increasing wavenumber magnitude. For wavenumbers less, in magnitude, than the free wavenumber of the string, similar arguments can be used to show that the applied force is primarily balanced by inertial forces in the string. These inertial forces are independent of wavenumber and act 180 degrees out of phase with both the tensile forces and the local displacement. Thus, in the wavenumber range $|k| < |\omega/c_s|$, where inertial forces dominate, the local displacement is nearly constant and out of phase with the applied force.

When the magnitude of the wavenumber of the applied force is in the neighborhood of, but not at, the free wavenumber of the string, the tensile and inertial forces in the string nearly cancel each other, and the displacement becomes very large. The relative phase between the displacement and the applied force in this wavenumber region is determined by the relative dominance of the tensile and inertial forces.

From the above discussion, the real part of the wavenumber frequency response defines the amplitude and relative phase of the displacement field

resulting from the unit amplitude harmonic force, $\exp\{i(kx + \omega t)\}$, at all wavenumbers and frequencies of the applied force except those characterized by $k = \omega/c_s$.

At the wavenumber-frequency combinations characterized by $k = \omega/c_s$, equation (3-82) and figure 3-5 show the real part of $G(k,\omega)$ to be undefined and the imaginary part to be a pair of weighted Dirac delta functions. The imaginary part, introduced by the wavenumber-frequency transform of the homogeneous part of the Green's function, characterizes free waves in the string. Recall that these free waves were necessary in order that the Green's function be causal. Regardless of the value of the real part of $G(k,\omega)$ at $k = \omega/c_s$, the delta functions in the imaginary part ensure an infinite displacement of the string when the steady state harmonic wave excitation coincides with a free wave: that is, $f(x,t) = \exp\{ik(x \pm c_s t)\}$ for any k. While this result is consistent with physical intuition, it is not possible, by this example alone, to physically interpret the separate roles of the real and imaginary parts of the wavenumber-frequency response for harmonic excitations coincident with free waves in the string.

With the above background, let us now examine the wavenumber-frequency description (i.e., transform), $W(k,\omega)$, of the space-time displacement field of the string, $w(x,t)$, resulting from an arbitrary force per unit length, $f(x,t)$, applied to the string. By equations (3-59) and (3-82), this wavenumber frequency transform is given by

$$W(k,\omega) = \frac{-F(k,\omega)}{\varepsilon[\omega^2 - (kc_s)^2]} + \frac{\pi F(k,\omega)}{2i\varepsilon kc_s}\{\delta(\omega - kc_s) - \delta(\omega + kc_s)\} , \quad (3\text{-}83)$$

where $F(k,\omega)$ is the wavenumber-frequency transform of $f(x,t)$.

Recall that $W(k,\omega)$ defines the complex amplitudes of the various harmonic waves of the form $\exp\{i(kx + \omega t)\}$ comprising the displacement field as a function of the wavenumber and frequency characterizing each wave. Equation (3-83) clearly shows that the displacement field is comprised of only those harmonic wave components present in the forcing field.

At all wavenumber-frequency combinations, except those that characterize free waves in the string (i.e., $|k| = |\omega/c_s|$), the amplitudes and initial phases of the wavenumber-frequency components of the displacement field are defined by the product of the wavenumber-frequency transform of the forcing function and the real part of the wavenumber-frequency response of the string. This product is equivalent to a filtering of the forcing function, in both wavenumber and frequency, by the real part of the wavenumber-frequency response. Figure 3-6(a) illustrates how the magnitudes of the various wavenumber components of the forcing function are filtered, at some fixed frequency, by the magnitude of the real part of the wavenumber-frequency response of the string. The product of these magnitudes is the magnitude of the complex amplitude of the harmonic wave components of the displacement field at the corresponding wavenumber and frequency. Figure 3-6(a) clearly shows that, at any frequency, the magnitude of $W(k,\omega)$ will be relatively large at (1) those wavenumbers where $F(k,\omega)$ is large and (2) in the neighborhood of $\pm\omega/c_s$, if $F(k,\omega)$ is nonzero in that wavenumber range.

Figure 3-6(b) illustrates the phase shift applied to the various wavenumber components of the forcing function (at the same fixed frequency) by the wavenumber-frequency response of the string. The initial phase of $W(k,\omega)$ at each wavenumber is determined by applying this phase shift to the phase of $F(k,\omega)$ at the corresponding wavenumber. Note that the phase at $k = \pm\omega/c_s$ is undefined.

It will be recalled that the real part of the wavenumber-frequency response of the string is undefined at all wavenumber-frequency combinations defined by $|k| = |\omega/c_s|$. It follows therefore, by equation (3-59), that $W(k,\omega)$ is undefined at any wavenumber and frequency where $|k| = |\omega/c_s|$ and $F(k,\omega)$ is nonzero.

The above example illustrates a technique for obtaining the wavenumber-frequency response by treating the forced vibration of an infinite, uniform string. This example further shows that, given the wavenumber-frequency transform of the applied force and the wavenumber-frequency response of the string, a description of the harmonic waves comprising the displacement field of the string can be determined and interpreted at all wavenumbers and

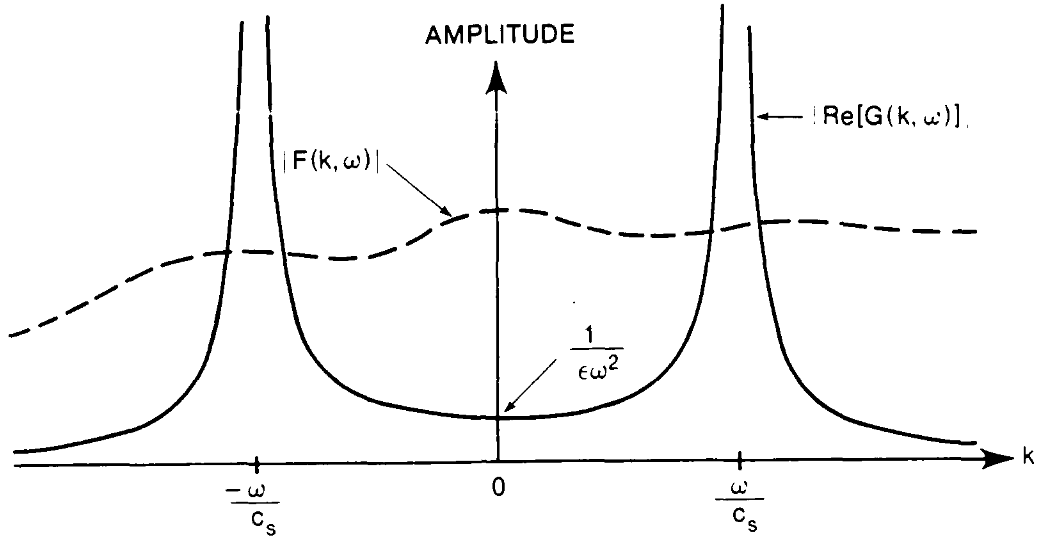

Figure 3-6(a). Filtering of the Magnitude of the Forcing Field by the Magnitude of the Real Part of the Wavenumber-Frequency Response

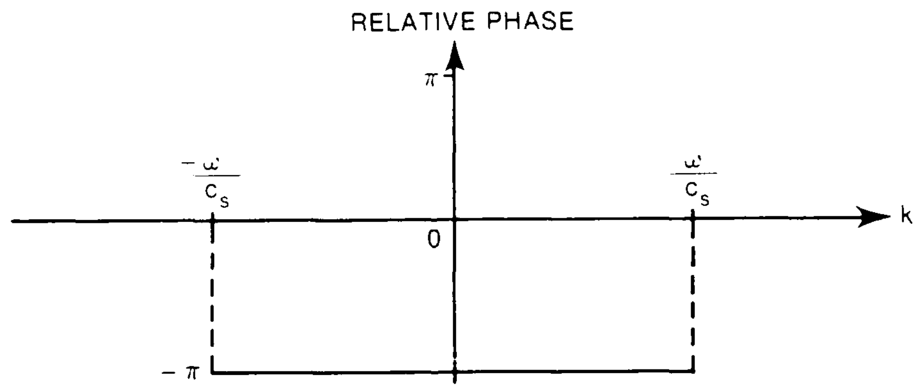

Figure 3-6(b). Phase of W(k,ω) Relative to F(k,ω)

Figure 3-6. Filtering of the Forcing Field by the Wavenumber-Frequency Response of the String

frequencies, except those coincident with free waves in the string, where the wavenumber-frequency response is undefined.

The reason that the wavenumber-frequency response is undefined at $|k| = |\omega/c_s|$ is the absence of any loss mechanism in the mathematical model of the string. As we will show in the next example, the inclusion of losses in the mathematical model permits definition of the harmonic waves comprising the displacement field at all wavenumbers and frequencies.

3.4.5 The Forced Vibration of a Damped, Infinite String

Consider the displacement of a damped, uniform, infinitely long string resulting from a force per unit length, f(x,t), applied to the string. The damping force per unit length opposes the motion of the string and is proportional to the local velocity. To ensure that this damping force is space and time invariant, we assume this proportionality (denoted by r) to be constant. The mathematical model of this system is given by

$$T \frac{\partial^2 w}{\partial x^2} - \varepsilon \frac{\partial^2 w}{\partial t^2} - r \frac{\partial w}{\partial t} = -f(x,t) \ , \tag{3-84}$$

where T and ε are, respectively, the constant tension and mass per unit length of the string.

The Green's function for the damped string is the solution to equation (3-84) when f(x,t) is replaced by $\delta(x - x_0)\delta(t - t_0)$. As equation (3-84) applies over all space and time, we let $\xi = x - x_0$ and $\tau = t - t_0$ and write the equation for the Green's function as

$$\frac{\partial^2 g}{\partial \xi^2} - \frac{1}{c_s^2} \frac{\partial^2 g}{\partial \tau^2} - \frac{r}{\varepsilon c_s^2} \frac{\partial g}{\partial \tau} = \frac{-\delta(\xi)\delta(\tau)}{\varepsilon c_s^2} \ , \tag{3-85}$$

where it will be recalled that $c_s^2 = T/\varepsilon$.

As we did for the undamped string, we assume the particular solution, g_p, of equation (3-85) can be written in the form

$$g_p(\xi,\tau) = (2\pi)^{-2} \iint_{-\infty}^{\infty} G_p(k,\omega)\exp\{i(k\xi + \omega\tau)\}\, dk\, d\omega . \tag{3-86}$$

Then, by use of equations (2-38) and (3-86) in equation (3-85), arguments identical to those used between equations (3-67) and (3-69) yield

$$G_p(k,\omega) = \frac{-1}{\varepsilon[\omega^2 - (kc_s)^2 - ir\omega/\varepsilon]} . \tag{3-87}$$

In anticipation of applying the initial conditions for the causal Green's function in the same form as we did for the undamped string, we wish to obtain the wavenumber transform of the particular solution $\tilde{G}_p(k,t)$ by the inverse temporal Fourier transformation of equation (3-87). It is straightforward to show, by partial fraction expansion of equation (3-87), that

$$\tilde{G}_p(k,t) = \frac{-1}{4\pi\varepsilon\omega_d(k)} \left\{ \int_{-\infty}^{\infty} \frac{\exp(i\omega\tau)\, d\omega}{\omega - ir/(2\varepsilon) - \omega_d(k)} - \int_{-\infty}^{\infty} \frac{\exp(i\omega\tau)\, d\omega}{\omega - ir/(2\varepsilon) + \omega_d(k)} \right\} , \tag{3-88}$$

where

$$\omega_d(k) = \sqrt{[kc_s]^2 - [r/(2\varepsilon)]^2} . \tag{3-89}$$

By reference 10, it can be shown that

$$\int_{-\infty}^{\infty} \frac{\exp(i\omega\tau)\, d\omega}{\omega - ir/(2\varepsilon) + \omega_d(k)} = 2\pi i U(\tau)\exp\{-[r/(2\varepsilon) + i\omega_d(k)]\tau\} . \tag{3-90}$$

It follows, by equations (3-88) and (3-90), that

$$\tilde{G}_p(k,\tau) = (1/\varepsilon)U(\tau)\exp\{-r\tau/(2\varepsilon)\} \frac{\sin\{\omega_d(k)\tau\}}{\omega_d(k)} . \tag{3-91}$$

Equation (3-91) is the wavenumber transform of the particular solution to equation (3-85).

To obtain the wavenumber transform of the homogeneous solution to equation (3-85), we assume that the homogeneous portion of the Green's function can be written in the form

$$g_h(\xi,\tau) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{G}_h(k,\tau)\exp\{ik\xi\}\, dk \ . \qquad (3\text{-}92)$$

By substituting equation (3-92) into the homogeneous form of equation (3-85) and by realizing that the resultant integral applies for all ξ, we obtain the ordinary differential equation

$$\frac{d^2\tilde{G}_h}{d\tau^2} + \frac{r}{\varepsilon}\frac{d\tilde{G}_h}{d\tau} + (kc_s)^2\, \tilde{G}_h = 0 \ . \qquad (3\text{-}93)$$

The solution to equation (3-93) is

$$\tilde{G}_h(k,\tau) = \exp\{-r\tau/(2\varepsilon)\}\{A(k)\exp[i\omega_d(k)\tau] + B(k)\exp[-i\omega_d(k)\tau]\} \ , \qquad (3\text{-}94)$$

where $\omega_d(k)$ is given by equation (3-89).

As we argued for the undamped string, the causality condition that $g(\xi,\tau) = 0$ for $\tau < 0$ at all ξ can be translated to the condition that $\tilde{G}(k,\tau) = 0$ for $\tau < 0$ at all k. Therefore, for a causal system, we require that $\tilde{G}_p(k,\tau) + \tilde{G}_h(k,\tau) = 0$ for $\tau < 0$. By equations (3-91) and (3-94), this condition can be satisfied only if $A(k) = B(k) = 0$, from which it follows that $\tilde{G}_h(k,\tau) = 0$ for all k.

Therefore,

$$\tilde{G}(k,\tau) = \tilde{G}_p(k,\tau) = (1/\varepsilon)U(\tau)\exp\{-r\tau/(2\varepsilon)\}\ \frac{\sin\{\omega_d(k)\tau\}}{\omega_d(k)} \qquad (3\text{-}95)$$

and

$$G(k,\omega) = G_p(k,\omega) = \frac{-1}{\varepsilon[\omega^2 - (kc_s)^2 - ir\omega/\varepsilon]} . \tag{3-96}$$

Equation (3-96), which defines the wavenumber-frequency response for the damped string, shows that the causal Green's function is completely defined by the particular solution to equation (3-85). This result is in contrast to the Green's function of the undamped string, where the inclusion of the homogeneous solution was necessary to satisfy causality.

Before examining the properties of the wavenumber-frequency response, it would be interesting to determine the causal Green's function of the damped, infinite string for comparison with the undamped case. By use of equation (3-89), it is evident that $G(k,\tau)$, in equation (3-95), is an even function of k. By use of reference 11 and the properties of the Heaviside function, one can perform the inverse Fourier transform of equation (3-95) to obtain

$$g(\xi,\tau) = [1/(2\varepsilon c_s)]U\{c_s\tau - |\xi|\}\exp\{-r\tau/(2\varepsilon)\}I_0\left\{[r/(2\varepsilon c_s)]\sqrt{(c_s\tau)^2 - \xi^2}\right\} , \tag{3-97}$$

where I_0 is the zero-th order, modified Bessel function of the first kind. Comparison of this result with the causal Green's function of the undamped string, given by equation (3-81), shows that the damping introduces a temporal decay (via the negative exponential) and a spatial decay (via the modified Bessel function) into the causal Green's function.

Figure 3-7 illustrates the spatial dependence of the causal Green's function of the infinite, damped string at a constant time, τ, after application of the impulsive loading. By comparison with figure 3-4, the obvious difference between the Green's functions of the damped and undamped strings is that the Green's function of the damped string decreases with increasing magnitude of ξ in the range $|\xi| < c_s\tau$, whereas that of the undamped string is constant in this range. Another difference, however, is that the amplitude of the Green's function for the undamped string is constant, whereas that for the damped string decreases with increasing τ.

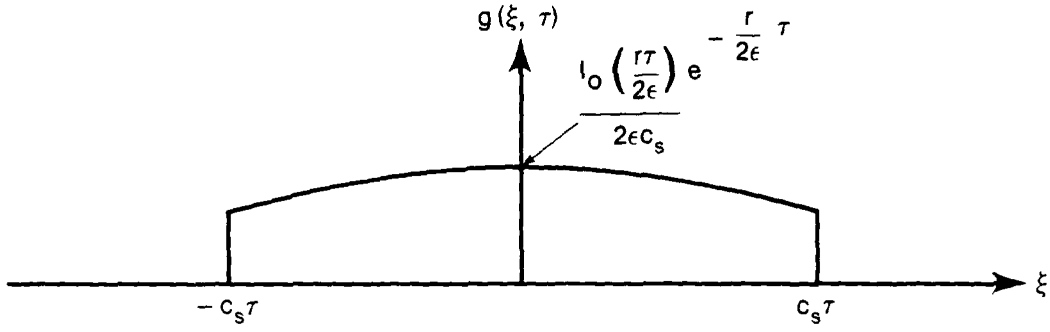

Figure 3-7. Spatial Dependence of the Causal Green's Function for a Damped, Infinite String

Let us now examine the characteristics of the wavenumber-frequency response. By a rearrangement of equation (3-96), we can separate $G(k,\omega)$ for the damped string into its real and imaginary parts as follows:

$$G(k,\omega) = \frac{-\{[\omega^2 - (kc_s)^2] + i[r\omega/\varepsilon]\}}{\varepsilon\{[\omega^2 - (kc_s)^2]^2 + [r\omega/\varepsilon]^2\}} \quad . \tag{3-98}$$

Figures 3-8(a) and (b) illustrate the wavenumber dependence of the real and imaginary parts of the wavenumber-frequency response of the damped string at a fixed frequency, ω, for a (constant) damping coefficient, r, such that the ratio $r/(\varepsilon\omega) = 0.1$.

Comparison of figure 3-8(a) with figure 3-5(a) shows the real parts of the wavenumber-frequency responses of the damped and undamped strings to be similar, except in the neighborhood of $k = \pm\omega/c_s$, where the response of the damped string remains defined, whereas that of the undamped string is undefined. In these regions of similarity, damping forces are insignificant, so the physical interpretation of the wavenumber-frequency response of the undamped string can be shown to apply to the damped string. At $k = \omega/c_s$, figure 3-8(a) shows the real part of $G(k,\omega)$ to be zero. Recall, from the discussion of the undamped string, that the tensile and inertial forces in the string are in balance at this wavenumber. Thus, in the damped string, the applied force at this wavenumber must be balanced by the forces due to

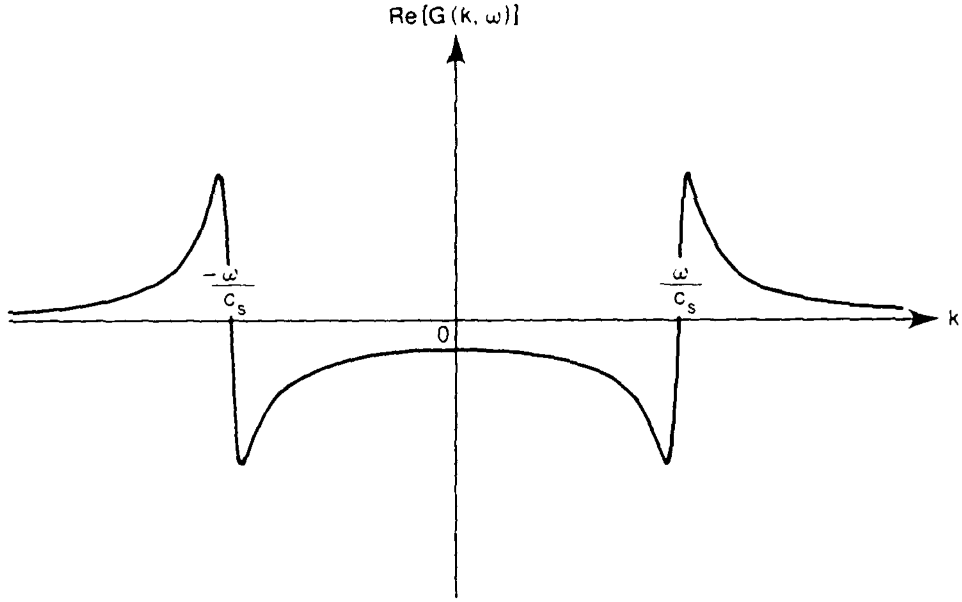

Figure 3-8(a). Real Part of Wavenumber-Frequency Response

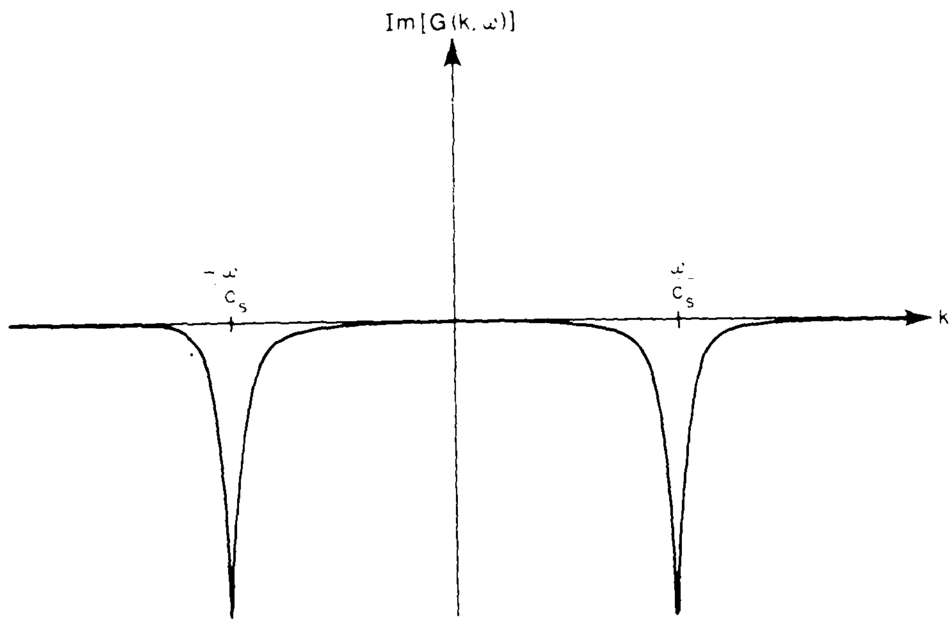

Figure 3-8(b). Imaginary Part of Wavenumber-Frequency Response

Figure 3-8. Real and Imaginary Parts of Wavenumber-Frequency Response of a Damped, Infinite String

damping. By inspection of equation (3-98), these damping forces are reflected in the imaginary part of $G(k,\omega)$.

Recall that the wavenumber-frequency response can be interpreted as the ratio of $w(x,t)$ to $f(x,t)$ when $f(x,t) = \exp\{i(kx + \omega t)\}$ for all x and t. By substitution of a solution of the form $w(x,t) = G(k,\omega)\exp\{i(kx + \omega t)\}$ into equation (3-84), it is easily established that $ir\omega$ is the ratio of the damping force per unit length to the displacement field, $w(x,t)$. Recall, from the example of the undamped string, that $\epsilon\omega^2$ and $\epsilon k^2 c_s^2$ are the ratios of the inertial and tensile forces per unit length, respectively, to the displacement field. For the fixed frequency and damping coefficient selected for this example, the ratio of the inertial force to the damping force is 10:1. With this background, the behavior of the imaginary part of $G(k,\omega)$, illustrated in figure 3-8(b), can easily be understood.

In the wavenumber ranges $|k| > |\omega/c_s|$, where tensile forces dominate both inertial and damping forces, equations (3-98) and figure 3-8(b) show the imaginary part of $G(k,\omega)$ to be small and negative. In the wavenumber range $|k| < |\omega/c_s|$, inertial forces dominate tensile forces. However, as stated above, the inertial forces are about 10 times greater than the damping forces, so, according to equation (3-98), the imaginary part of $G(k,\omega)$ is nearly constant and about 10 times smaller than the real part of $G(k,\omega)$ in this wavenumber range. At wavenumbers in the neighborhood of $|\omega/c_s|$, where the tensile and inertial forces nearly balance, the imaginary part of $G(k,\omega)$ is dictated by the ratio of the displacement to the damping force. As the inertial and tensile forces come into balance, the applied force must be balanced by the damping force. For the typically small damping coefficient used in this example, the displacement must be large to produce a force equal to the applied force. Hence, the ratio of the displacement to the damping force per unit length is large in the neighborhood of $|k| = |\omega/c_s|$. This behavior is reflected in the imaginary part of $G(k,\omega)$ depicted in figure 3-8(b).

As a final observation, note that at $|k| = |\omega/c_s|$,

$$G(|\omega/c_s|,\omega) \approx \mathrm{Im}[G(|\omega/c_s|,\omega)] = -i/(r\omega) , \qquad (3\text{-}99)$$

where Im[] denotes the imaginary part. Thus, as the damping (dictated by r) decreases, the imaginary part of $G(k,\omega)$ tends to infinity at $|k| = |\omega/c_s|$. Further, as the damping coefficient decreases, equation (3-98) can be used to show that the width, in wavenumber, of the negative peaks at $k = \pm\omega/c_s$ in the imaginary part of $G(k,\omega)$ decreases, while the amplitudes of the positive and negative peaks on either side of $k = \pm\omega/c_s$ in the real part of $G(k,\omega)$ increase. Thus, in the limit, as r tends to zero, the real and imaginary parts of $G(k,\omega)$ tend toward those shown for the undamped string. Further, by this limiting process, the behavior of the real part of $G(k,\omega)$ remains interpretable as the damping tends to zero. The lesson here is that undamped systems are best understood and interpreted when they are treated as limiting cases of damped systems.

For the general case of the forced, infinite, damped string, the magnitude and initial phase of each of the various harmonic waves comprising the space-time displacement field can be obtained as a function of the wavenumber and frequency characterizing each wave. We first write

$$
\begin{aligned}
W(k,\omega) &= |W(k,\omega)| \exp[i\alpha(k,\omega)] , \\
F(k,\omega) &= |F(k,\omega)| \exp[i\beta(k,\omega)] , \qquad (3\text{-}100) \\
G(k,\omega) &= |G(k,\omega)| \exp[i\sigma(k,\omega)] ,
\end{aligned}
$$

where $F(k,\omega)$ is the wavenumber-frequency transform of the applied forcing field per unit length, $f(x,t)$; $\alpha(k,\omega)$ and $\beta(k,\omega)$ are the initial phases of the harmonic waves comprising $w(x,t)$ and $f(x,t)$, respectively, as a function of the wavenumber and frequency characterizing each wave; and $\sigma(k,\omega)$ is the argument of $G(k,\omega)$. By use of equations (3-59) and (3-100), it follows that the magnitude of $W(k,\omega)$ is equal to the magnitude of $F(k,\omega)$ filtered by the magnitude of $G(k,\omega)$. That is,

$$
|W(k,\omega)| = |F(k,\omega)G(k,\omega)| = |F(k,\omega)|\;|G(k,\omega)| . \qquad (3\text{-}101)
$$

It also follows that the initial phase, $\alpha(k,\omega)$, of each complex harmonic wave comprising $w(x,t)$ is equal to the initial phase, $\beta(k,\omega)$, of the corresponding wave component of $f(x,t)$ shifted by the argument of $G(k,\omega)$. That is,

$$\alpha(k,\omega) = \beta(k,\omega) + \sigma(k,\omega) \ . \tag{3-102}$$

Figures 3-9(a) and (b) depict the filtering of the magnitude of $F(k,\omega)$ by the magnitude of $G(k,\omega)$ and the phase shift, $\sigma(k,\omega)$, respectively, for the infinite, damped string. By equation (3-101), the magnitude of $W(k,\omega)$ will be large when the product of the magnitudes of $F(k,\omega)$ and $G(k,\omega)$ are large. From figure 3-9(a), the magnitude of $G(k,\omega)$ has relative maxima at $k = \pm\omega/c_s$. Thus, unless $|F(k,\omega)|$ is small in this region, $|W(k,\omega)|$ will exhibit relative maxima at wavenumbers characterizing free waves in the string. Other relative maxima of $|W(k,\omega)|$ can occur, at any frequency, in the neighborhood of those wavenumbers characterizing large relative contributions to $|F(k,\omega)|$. Through this filtering process, the relative amplitudes of the various harmonic waves comprising the displacement field are determined as a function of the wavenumber and frequency characterizing each wave.

Figure 3-9(b) shows that the phase shift applied to each wavenumber component of the forcing field, at a fixed frequency, by the wavenumber-frequency response of the string is (1) small for $|k|$ large in comparison to $|\omega/c_s|$, (2) $-\pi/2$ at $|k| = |\omega/c_s|$, and (3) approximately $-\pi$ for $|k| < |\omega/c_s|$.

As a final comment, it should be noted that the magnitude and phase of $G(k,\omega)$ for the infinite, damped string, shown in figure 3-9, do not exhibit the discontinuities or undefined response in the vicinity of the free wavenumber found (see figure 3-6) in the magnitude and phase of $G(k,\omega)$ for the case of the undamped string. Thus, if one is interested in the response of a system near such resonances, it is clear that some estimate of the damping or loss must be included in the mathematical model of the system.

3.4.6 The Wavevector-Frequency Response of a Damped, Infinite Plate

This final example is included to demonstrate that the mathematical techniques employed to obtain the Green's function (or its informational equivalent, the wavenumber-frequency response) for the systems illustrated above, which depend on only one spatial variable, can be applied to systems requiring two or three independent spatial variables in their mathematical

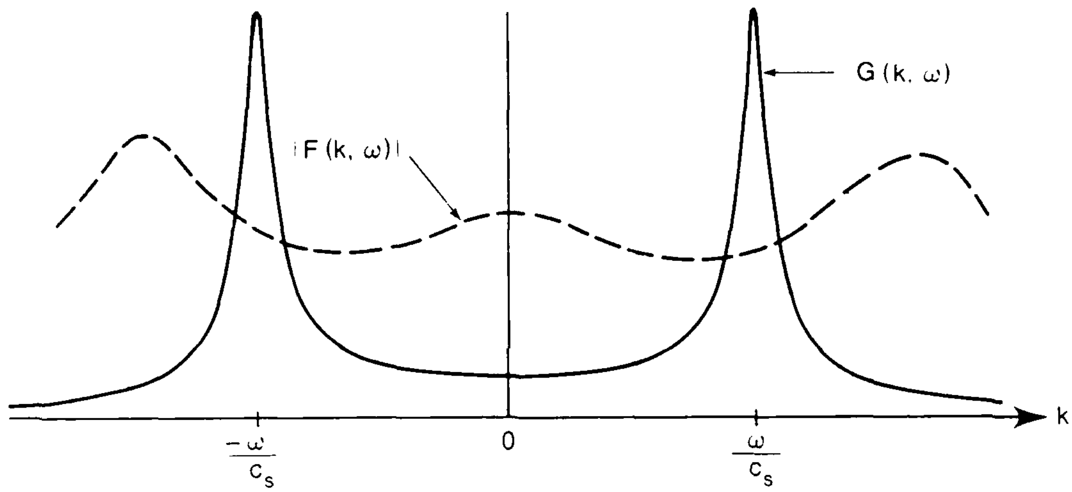

Figure 3-9(a). Filtering of $F(k,\omega)$ by $G(k,\omega)$

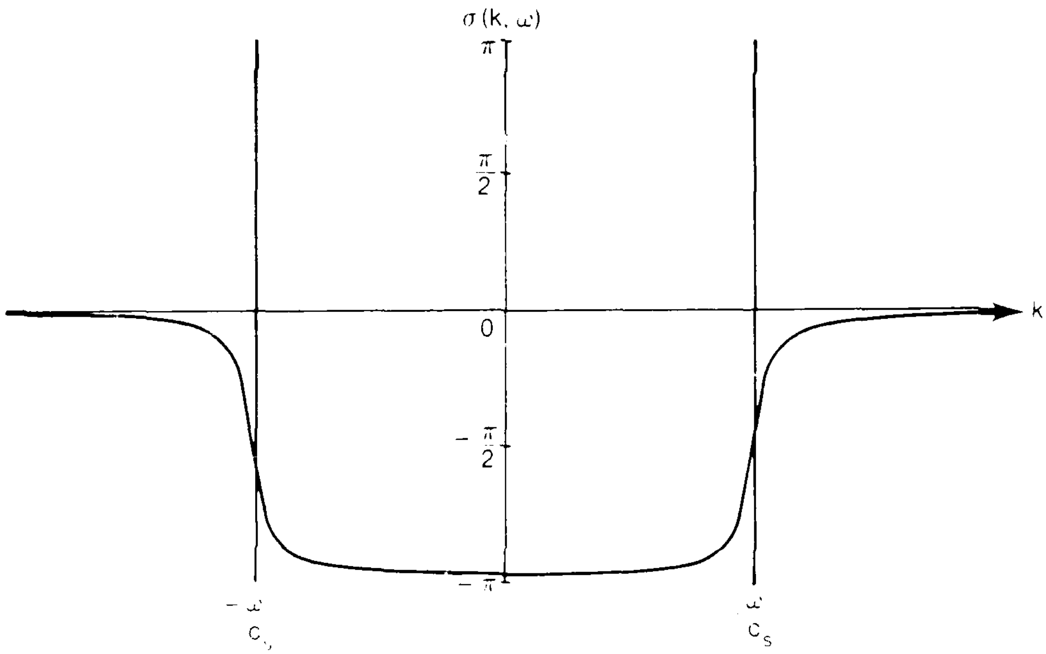

Figure 3-9(b). The Phase Shift, $\sigma(k,\omega)$

Figure 3-9. Filtering and Phase Shift of the Forcing Field by the Wavenumber Frequency Response of a Damped, Infinite String

model. The specific problem treated here is the wavevector-frequency response of a uniform, infinite, damped plate.

A mathematical model for the forced vibrations of a damped plate is given by Davies.[12] That model, in the notation adopted in section 3.3.2, states that

$$D\nabla^4 w(\underline{x},t) + r\frac{\partial w(\underline{x},t)}{\partial t} + \mu\frac{\partial^2 w(\underline{x},t)}{\partial t^2} = f(\underline{x},t) , \qquad (3\text{-}103)$$

where, for this space- and time-invariant system, the damping coefficient, r, is assumed constant and $f(\underline{x},t)$ is the force per unit area applied to the plate.

The mathematical model for the Green's function is obtained by replacing the applied force per unit area in equation (3-103) by $\delta(\underline{x} - \underline{x}_0)\delta(t - t_0)$, where $\delta(\underline{x} - \underline{x}_0) = \delta(x_1 - x_{01})\delta(x_2 - x_{02})$. However, because the infinite plate system is space- and time-invariant, we can define $\underline{\xi} = \underline{x} - \underline{x}_0 = [x_1 - x_{01}, x_2 - x_{02}]$ and $\tau = t - t_0$ and then write the equation governing the Green's function as

$$D\nabla^4 g(\underline{\xi},\tau) + r\frac{\partial g(\underline{\xi},\tau)}{\partial \tau} + \mu\frac{\partial^2 g(\underline{\xi},\tau)}{\partial \tau^2} = \delta(\underline{\xi})\delta(\tau) . \qquad (3\text{-}104)$$

As we did in the case of the forced vibration of the string, we assume a particular solution, $g_p(\underline{\xi},\tau)$, exists of the form

$$g_p(\underline{\xi},\tau) = (2\pi)^{-3}\iint_{-\infty}^{\infty} G_p(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{\xi} + \omega\tau)\}\, d\underline{k}\, d\omega . \qquad (3\text{-}105)$$

By equations (3-104), (3-105), and (2-38) and arguments similar to those employed between equations (3-67) and (3-69), it can be shown that

$$G_p(\underline{k},\omega) = \frac{1}{Dk^4 - \mu\omega^2 + ir\omega} , \qquad (3\text{-}106)$$

where $k = \sqrt{k_1^2 + k_2^2}$.

We argue here, as we did in the case of the vibration of the string, that the causality condition that $g(\underline{\xi},\tau) = 0$ for $\tau < 0$ at all $\underline{\xi}$ translates, under wavevector transformation, to $\tilde{G}(\underline{k},\tau) = 0$ for $\tau < 0$ at all wavevectors, $\underline{k}$.

By a partial fraction expansion of equation (3-106), the particular part of $\tilde{G}(\underline{k},\tau)$ can be written as

$$\tilde{G}_p(\underline{k},\tau) = \frac{1}{4\pi\mu\omega_d(k)} \left\{ \int_{-\infty}^{\infty} \frac{\exp(i\omega\tau)\ d\omega}{\omega - ir/(2\mu) + \omega_d(k)} - \int_{-\infty}^{\infty} \frac{\exp(i\omega\tau)\ d\omega}{\omega - ir/(2\mu) - \omega_d(k)} \right\} , \tag{3-107}$$

where

$$\omega_d(k) = \sqrt{[Dk^4/\mu] - [r/(2\mu)]^2} \ . \tag{3-108}$$

Comparison of equation (3-107) with equation (3-88) reveals that the particular part of $\tilde{G}(\underline{k},\tau)$ for the infinite, damped plate has the same mathematical form as the particular part of $\tilde{G}(\underline{k},\tau)$ for the infinite, damped string. It therefore follows, by the arguments of equations (3-90) and (3-91), that

$$\tilde{G}_p(\underline{k},\tau) = (1/\mu)U(\tau)\exp[-r\tau/(2\mu)]\ \frac{\sin\{\omega_d(k)\tau\}}{\omega_d(k)} \ . \tag{3-109}$$

The homogeneous solution, g_h, to equation (3-104) is assumed to exist in the form

$$g_h(\underline{\xi},\tau) = (2\pi)^{-2} \int_{-\infty}^{\infty} \tilde{G}_h(\underline{k},\tau)\exp\{i\underline{k}\cdot\underline{\xi}\}\ d\underline{k} \ . \tag{3-110}$$

By substituting equation (3-110) into the homogeneous form of equation (3-104) and by realizing that the resultant integral applies for all $\underline{\xi}$, we obtain the ordinary differential equation

$$\mu \frac{d^2\tilde{G}_h}{d\tau^2} + r \frac{d\tilde{G}_h}{d\tau} + Dk^4\tilde{G}_h = 0 \ . \tag{3-111}$$

The solution to equation (3-111) is easily shown to be

$$\tilde{G}_h(\underline{k},\tau) = \exp\{-r\tau/(2\mu)\}\{A(\underline{k})\exp[i\omega_d(k)\tau] + B(\underline{k})\exp[-i\omega_d(k)\tau]\} \ , \tag{3-112}$$

where $\omega_d(k)$ is given by equation (3-108).

By the arguments given previously, the functions $A(\underline{k})$ and $B(\underline{k})$ are selected to satisfy causality; that is

$$\tilde{G}(\underline{k},\tau) = \tilde{G}_p(\underline{k},\tau) + \tilde{G}_h(\underline{k},\tau) = 0 \ , \quad \tau < 0 \ , \tag{3-113}$$

for all $\underline{k}$. By equations (3-109) and (3-112), it is evident that equation (3-113) can be satisfied only if $A(\underline{k}) = B(\underline{k}) = 0$. It follows that $G_h(k,\tau) = 0$ and therefore

$$G(\underline{k},\omega) = G_p(\underline{k},\omega) = \frac{1}{Dk^4 - \mu\omega^2 + ir\omega} \ . \tag{3-114}$$

A significant feature of the wavevector-frequency response of the infinite, damped plate, described by equation (3-114), is that it depends only on the magnitude of the wavevector ($\underline{k}$) and not on its direction. Inasmuch as $G(\underline{k},\omega)$ is the ratio of the space-time displacement field, $w(\underline{x},t)$, to the forcing field, $f(\underline{x},t)$, when the forcing field is given by $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ for all $\underline{x}$ and t, it follows that $w(\underline{x},t) = G(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$. Thus, $G(\underline{k},\omega)$ can also be interpreted as the complex amplitude of the wave of displacement of the plate that corresponds, in wavevector and frequency, to the wave that excites the plate in motion. By the arguments of chapter 2, knowledge of the frequency and wavevector magnitude determines the wavelength and period of a (complex) plane harmonic wave. The direction of propagation is determined by the direction of the wavevector and the sign of the frequency. Therefore, the dependence of $G(\underline{k},\omega)$ on only the magnitude of the wavevector can be

interpreted as a reflection of the spatial invariance, or isotropy, of the plate. That is, for a unit amplitude, harmonic wave excitation of the plate, the amplitude of the resultant displacement of the plate depends only on the wavelength and frequency of the excitation and is independent of the direction of propagation of the harmonic wave excitation.

Figures 3-10(a) and (b) illustrate the magnitude and phase of the wavevector-frequency response of the damped, infinite plate as a function of wavevector magnitude, k, at an arbitrary, fixed, positive frequency. At this frequency, the (constant) damping coefficient was taken to be $r = 0.1\ \mu\omega$.

The behavior of the magnitude and phase of $G(\underline{k},\omega)$ with k, depicted in figure 3-10, is easily understood by recalling that for the harmonic wave excitation $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$, the displacement field is given by $w(\underline{x},t) = G(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$. Substitution of these displacement and excitation fields into equation (3-103) reveals that for $k < k_p(\omega) = \sqrt[4]{\mu\omega^2/D}$, the excitation is primarily balanced by the inertial forces in the plate and $G(\underline{k},\omega) = w(\underline{x},t)/f(\underline{x},t) \approx 1/(-\mu\omega^2)$. Thus, in this wavenumber range, the magnitude of G is nearly constant with k, and the displacement is nearly 180 degrees out of phase with the applied force.

In the wavenumber range $k > k_p(\omega)$, the applied force is primarily balanced by the forces associated with the bending stiffness of the plate, given by $Dk^4 w(\underline{x},t)$. Thus, in this wavenumber range, it follows that $G(\underline{k},\omega) = w(\underline{x},t)/f(\underline{x},t) \approx 1/(Dk^4)$. Here, therefore, the magnitude of G decreases with increasing wavevector magnitude as k^{-4}, and, as depicted in figure 3-10(b), the displacement is nearly in phase with the applied force.

In the neighborhood of $|\underline{k}| = k_p(\omega)$, the forces associated with bending stiffness and inertia nearly cancel each other, and the applied forces are primarily balanced by the damping force, given by $ir\omega w(\underline{x},t)$. The damping coefficient, r, was chosen such that, at the fixed frequency of this example, the damping force was one-tenth of the inertial force. Thus, at $|\underline{k}| = k_p(\omega)$, the magnitude of G is about 10 times greater than it is in the wavenumber range $|\underline{k}| < k_p(\omega)$, where inertial forces dominate. The initial phase of $w(\underline{x},t)$, at $|\underline{k}| = k_p(\omega)$, is seen to lag that of the harmonic excitation by 90 degrees.

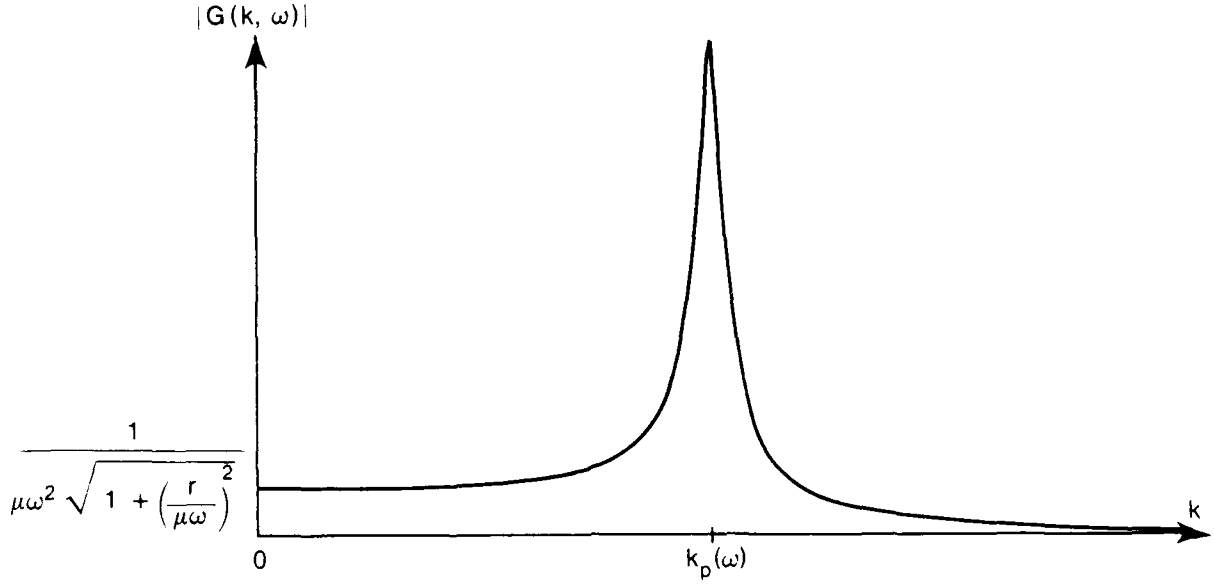

Figure 3-10(a). Magnitude of Wavevector-Frequency Response

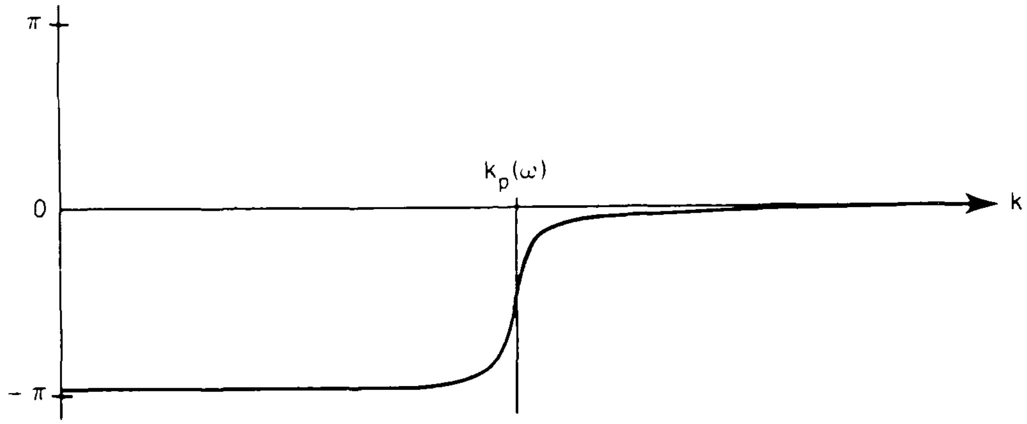

Figure 3-10(b). Phase of Wavevector-Frequency Response

Figure 3-10. Magnitude and Phase of the Wavevector-Frequency Response of a Damped, Infinite Plate

In figure 3-10, we plotted the magnitude and phase of the wavevector-frequency response as a function of k (i.e., the magnitude of the wavevector, $\underline{k}$) at an arbitrary frequency for purposes of graphical convenience. However, if one wished to use the relation $W(\underline{k},\omega) = F(\underline{k},\omega)G(\underline{k},\omega)$ to determine $W(\underline{k},\omega)$ for an arbitrary excitation of the plate, $F(\underline{k},\omega)$, it must be realized that both the magnitude and phase of $G(\underline{k},\omega)$ are circularly symmetric functions in the (k_1,k_2) plane. To illustrate this circular pattern of the wavevector-frequency response in the $\underline{k}$ plane, figure 3-11 shows the locus of the maximum magnitude of $G(\underline{k},\omega)$ for the damped, infinite plate in the $\underline{k}$ plane at an arbitrary, fixed frequency. As shown in figure 3-10(a) and illustrated in figure 3-11, the magnitude of $G(\underline{k},\omega)$ has a maximum at those wavevectors having magnitudes equal to the free wavenumber of the plate at the frequency of interest: that is, at $k = \sqrt{k_1^2 + k_2^2} = k_p(\omega)$, where $k_p(\omega) = \sqrt[4]{\mu\omega^2/D}$.

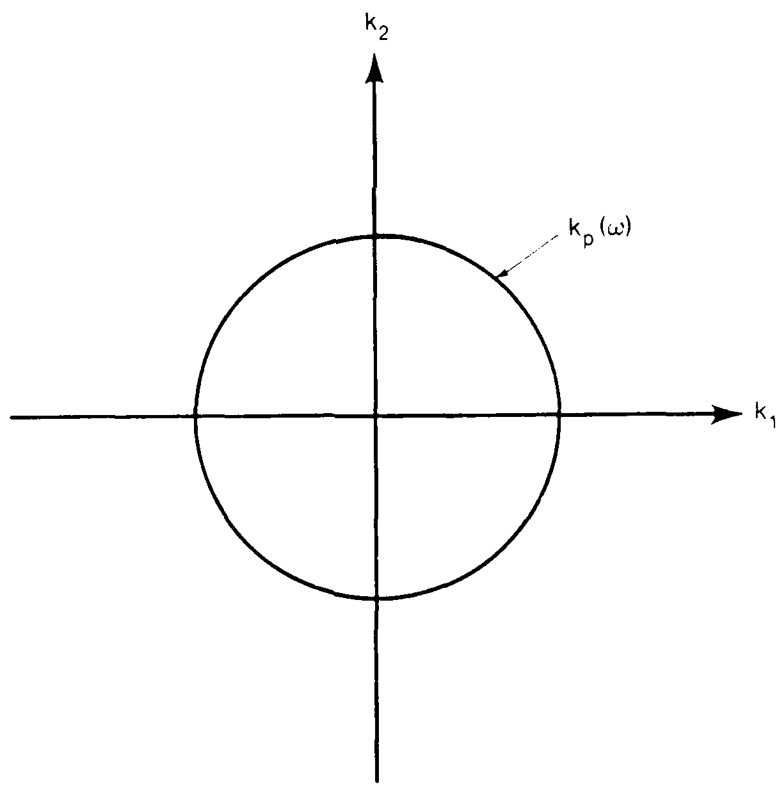

Figure 3-11. Locus of the Maximum Magnitude of $G(\underline{k},\omega)$ for a Damped, Infinite Plate

3.4.7 Summary of the Forced Response of Space- and Time-Invariant Linear Systems

The approach to the forced response of linear systems taken in this text is that of linear superposition by use of the Green's function. The Green's function is defined as the response of the linear system, in space and time, to an impulsive loading applied at any arbitrary location in space and at any arbitrary time. Inasmuch as (1) any input can be expressed as a weighted integral (summation) of impulses (i.e., Dirac delta functions) in space and time and (2) any summation of solutions to the linear differential equation describing a system also constitutes a solution to that equation, it then follows that the response of the system to an arbitrary input can be expressed as an integral of the Green's function weighted by the space-time excitation field.

For space- and time-invariant systems, the coefficients of the linear differential equations governing the system are constants. As a consequence, the Green's functions of linear space- and time-invariant systems depend only on the spatial separation vector, $\vec{\xi}$, and the temporal difference, τ, between the coordinates of observation, $(\vec{x},t)$, and excitation, $(\vec{x} - \vec{\xi}, t - \tau)$. As a result, the relation between the output field, $p(\vec{x},t)$, the input field, $f(\vec{x},t)$, and the Green's function, $g(\vec{\xi},\tau)$, is the convolution given by equation (3-52). This useful result states that, given the Green's function of a space- and time-invariant linear system, the output field of that system resulting from any input field can, in principle, be predicted.

By a Fourier transformation of the Green's function solution on all space and time variables, a simple algebraic expression is obtained that relates the wavevector-frequency transform of the output field to the product of the transforms of the input field and the Green's function. This relation is given by equation (3-59). By this equation, the amplitudes and initial phases of each plane harmonic wave component of the output field can be obtained, given the wavevector-frequency transform of the Green's function, $G(\vec{k},\omega)$, and the amplitudes and initial phases of the corresponding wave components of the input. The wavevector-frequency transform of the Green's function is called the wavenumber-frequency response of the system because it can be shown to be

equal to the ratio of the space-time output field to the space-time input field when that input field is the single complex harmonic plane wave $\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}$ for all $\vec{x}$ and t.

Systems of practical interest in acoustics are causal systems. That is, they respond only to past inputs and do not respond in anticipation of future inputs. Therefore, in deriving the Greens' function or its informational equivalent, the wavevector-frequency response, it is important to ensure that the Green's function satisfies conditions of causality specified by equation (3-53).

To illustrate the mathematical techniques for obtaining causal Green's functions or wavevector-frequency responses and to demonstrate how the wavevector-frequency description of forced space- and time-invariant systems can be physically interpreted, illustrative examples are presented in sections 3.4.4, 3.4.5, and 3.4.6.

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CHAPTER 4
SPACE-VARYING LINEAR SYSTEMS

4.1 INTRODUCTION

The spatially distributed, continuous time systems of linear acoustics are mathematically modeled by partial differential equations of the form of equation (3-37): that is, by

$$L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t) \ . \qquad (4\text{-}1)$$

Here $f(\vec{x},t)$ is the system input, $p(\vec{x},t)$ is the system output, and $L_{\vec{x},t}\{\ \}$ is the partial differential operator defined (see equation (3-36)) by

$$L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J}\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} a_{jlmn}(\vec{x},t)\left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} . \qquad (4\text{-}2)$$

The space- and time-invariant linear systems treated in chapter 3 were defined as systems having constant properties, or parameters, over all space and time. In the mathematical model of acoustic systems specified by equations (4-1) and (4-2), system parameters are reflected in the coefficients, $a_{jlmn}(\vec{x},t)$, of the partial differential operator, $L_{\vec{x},t}\{\ \}$. Thus, for space- and time-invariant linear systems, we required that

$$a_{jlmn}(\vec{x},t) = a_{jlmn} = \text{constant} \qquad (4\text{-}3)$$

for all $\vec{x}$ and t. By defining ${}_I L_{\vec{x},t}\{\ \}$ as that form of $L_{\vec{x},t}\{\ \}$ in which the parameters $a_{jlmn}(\vec{x},t)$ are constants over all space and time (that is,

$${}_I L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J}\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} a_{jlmn}\left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} \qquad (4\text{-}4)$$

for all $\vec{x}$ and t), it follows that space- and time-invariant linear acoustic systems can be mathematically modeled by

$$_{I}L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t) \tag{4-5}$$

for all $\vec{x}$ and t.

In this chapter, we explore the wavenumber-frequency analysis of space-varying, but time-invariant, linear acoustic systems. Space-varying, time-invariant systems are those in which the parameters of the system vary in space, but not in time. As we argued above, the system parameters are reflected in the general mathematical model of the linear acoustic system (equations (4-1) and (4-2)) by the coefficients $a_{jlmn}(\vec{x},t)$ contained in the linear partial differential operator, $L_{\vec{x},t}\{\ \}$. Therefore, if the coefficients a_{jlmn} in equation (4-2) describe the variation of the system parameters over all space and are constant in time (that is,

$$a_{jlmn}(\vec{x},t) = b_{jlmn}(\vec{x}) \tag{4-6}$$

for all $\vec{x}$ and t), we can define the space-varying, time-invariant linear partial differential operator, $_{It}L_{\vec{x},t}\{\ \}$, by

$$_{It}L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J}\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} b_{jlmn}(\vec{x})\left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} \tag{4-7}$$

for all space and time. If $_{It}L_{\vec{x},t}\{\ \}$ is substituted for $L_{\vec{x},t}\{\ \}$ in equation (4-1), it follows that space-varying, time-invariant systems in linear acoustics are mathematically modeled by

$$_{It}L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t) \tag{4-8}$$

for all $\vec{x}$ and t.

The mathematical form of equation (4-8) is an n th order linear partial differential equation with nonconstant coefficients. Finite (or closed) form solutions cannot be obtained for most ordinary linear differential equations

greater than first order with nonconstant coefficients.[1] Therefore, our exploration of space-varying, time-invariant acoustic systems cannot be general. Rather, the primary emphasis in this chapter will be to obtain wavevector-frequency descriptions of space-varying versions of some of the physical systems treated in chapter 3. Comparison of the wavevector-frequency descriptions of the outputs of space-varying and space-invariant versions of the system will then be used to illustrate the effects of the spatial variation. Another objective of this chapter is to develop and interpret wavevector-frequency descriptions of certain space-varying fields that arise in structural-acoustics.

Space-varying systems result from only two characteristics of the physical system: (1) boundaries and (2) nonuniformities in the spatial properties between elements of the system. Space-varying systems can therefore be classified according to the source of the spatial variation. Bounded systems are referred to as space limited, whereas unbounded systems are infinite. Systems are termed spatially uniform or nonuniform, depending on the respective absence or presence of spatial nonuniformities in properties over the physical extent of the system. By use of these definitions, it follows that space-varying systems can be classified into three categories: (1) uniform, space limited, (2) nonuniform, space limited, and (3) nonuniform, infinite. Uniform, infinite systems are, of course, space invariant.

Of the three categories of space-varying systems, the one most commonly encountered, and best understood, in acoustic applications is the uniform, space-limited system. Consequently, the primary focus in this chapter will be on the wavevector-frequency characteristics of uniform, space-limited acoustic systems.

As we did for space- and time-invariant systems, we will first present some illustrative examples of free, space-varying systems and then explore the forced response.

4.2 FREE RESPONSE OF SPACE-VARYING, TIME-INVARIANT SYSTEMS

Recall that free systems are systems free of externally imposed inputs, characterized by $f(\vec{x},t)$ in equation (4-8). Thus, free, space-varying,

time-invariant acoustic systems are modeled by equations of the form

$$I_t L_{\vec{x},t}\{p(\vec{x},t)\} = 0 . \tag{4-9}$$

Recall further, from the previous chapter, that free systems with losses cannot be described in the wavevector-frequency domain. Therefore, the illustrative examples presented below will be confined to lossless systems.

4.2.1 The Finite String With Fixed Ends

Consider the free vibration of a string of length L, fixed at $x = 0$ and $x = L$ such that no motion occurs at the ends. The tension, T, and mass per unit length, ε, are constant over the length of the string, $0 < x < L$, and are taken to be zero outside this interval. The equation governing the displacement field of the string, $w(x,t)$, can then be written as

$$b(x)\ T\left\{\frac{\partial^2 w}{\partial x^2} - \varepsilon\ \frac{\partial^2 w}{\partial t^2}\right\} = 0 \tag{4-10}$$

for all x and t. The spatially varying coefficient, $b(x)$, that modifies T and ε is defined by

$$b(x) = U(x) - U(x - L) = \begin{cases} 1\ ,\ 0 < x < L\ , \\ 0\ ,\ \text{otherwise}\ , \end{cases} \tag{4-11}$$

where $U(x)$ denotes the Heaviside function.

The requirement that the ends of the string be motionless translates into the boundary conditions

$$w(0,t) = w(L,t) = 0 \tag{4-12}$$

for all t. To complete the statement of the free vibration problem, we assume (as we did in the case of the uniform, infinite string) that the initial displacement and velocity fields of the string are given by

$$w(x,0) = w_0(x) \tag{4-13}$$

and

$$\frac{\partial w(x,0)}{\partial t} = v_0(x) \ . \qquad (4\text{-}14)$$

Equation (4-10) is equivalent to the mathematical statement

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c_s^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad (4\text{-}15)$$

for $0 < x < L$ and all t. Recall that $c_s^2 = T/\varepsilon$. The space-limited nature of this equation precludes a solution by the Fourier transform technique used for the free vibration of the infinite string in chapter 3. That is, although we may write the vibration field in the form

$$w(x,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{W}(k,t)\exp(ikx)\, dk \ , \qquad (4\text{-}16)$$

substitution of this form into equation (4-15) yields

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \left\{\frac{d^2\tilde{W}(k,t)}{dt^2} + (kc_s)^2\, \tilde{W}(k,t)\right\} \exp(ikx)\, dk = 0 \ , \qquad (4\text{-}17)$$

which is valid only over the spatial range $0 < x < L$. Thus, we cannot argue, as we did for the case of the free, infinite string, that

$$\frac{d^2\tilde{W}(k,t)}{dt^2} + (kc_s)^2\, \tilde{W}(k,t) = 0 \ , \qquad (4\text{-}18)$$

because equation (4-18) is a valid conclusion from equation (4-17) only when equation (4-17) holds for all x.

From the argument presented above, it is evident that the wavenumber-frequency description of the vibration field of the free, space-limited string cannot be obtained by direct application of Fourier transforms to the partial differential equation governing the motion of the string. However, the free

vibration of the finite-length string, fixed at its ends, is a classic problem in acoustics, and the space-time description of the displacement field for this system is derived in most standard texts on acoustics. Our approach, therefore, to obtaining a wavenumber-frequency description of the free vibration of the finite, fixed-end string is simply to perform a double Fourier transform of the classical solution for the space-time displacement field.

The space-time displacement field of the free, finite string with fixed ends is generally obtained (see reference 2, for example) by a separation of variables approach. This approach leads to a description of the displacement field, w(x,t), in terms of a complete set of orthogonal functions, called normal modes, in the variable x. Associated with each normal mode is a natural frequency. The details of this solution procedure for the space-time field of the finite string are well documented (see references 2 and 3, for example) and will not be further reviewed here.

From reference 3, the space-time displacement field of the finite string that satisfies the fixed-end boundary conditions of equation (4-12) and the arbitrary initial conditions specified by equations (4-13) and (4-14) is given by

$$w(x,t) = \sum_{n=1}^{\infty} \left\{ C_n \cos(\omega_n t) + D_n \sin(\omega_n t) \right\} \alpha_n(x) \tag{4-19}$$

for $0 \leq x \leq L$ and for all time. Here, the normal modes, denoted by $\alpha_n(x)$ and given by

$$\alpha_n(x) = \sin(n\pi x/L) \ , \tag{4-20}$$

form a complete set of orthogonal functions over the spatial interval $0 \leq x \leq L$. The orthogonality condition is given by

$$\int_0^L \alpha_m(x)\alpha_n(x)\, dx = (L/2)\delta_{mn} \ , \tag{4-21}$$

where δ_{mn} is the Kronecker delta. The modal natural frequencies, ω_n, are given by

$$\omega_n = n\pi c_s/L \ . \tag{4-22}$$

The modal coefficients, C_n and D_n, in equation (4-19) are determined by the modal content of the initial displacement and velocity. That is,

$$C_n = (2/L) \int_0^L w_0(x)\alpha_n(x)\ dx \tag{4-23}$$

and

$$D_n = 2/(n\pi c_s) \int_0^L v_0(x)\alpha_n(x)\ dx \ . \tag{4-24}$$

As is evident, by equation (4-19), the space-time displacement field of the free, finite, fixed-end string is expressed as a weighted sum of natural modes of vibration of the string, where each natural mode of vibration is characterized by a specific spatial pattern of displacement, $\alpha_n(x)$, and a specific frequency of vibration, ω_n. The amplitude and initial phase of each modal contribution to the displacement field is determined (see equations (4-23) and (4-24)) by the initial displacement and velocity fields of the string.

The wavenumber-frequency description, $W(k,\omega)$, of the displacement field is defined as the double Fourier transform of the space-time displacement field, $w(x,t)$, over all space and time. That is,

$$W(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,t)\exp\{-i(kx + \omega t)\}\ dx\ dt \ . \tag{4-25}$$

Equation (4-19) defines the space-time displacement field of the fixed-end, finite string only over the spatial interval $0 \le x \le L$. Outside this spatial

interval, the string does not exist, so the displacement in the regions $x < 0$ and $x > L$ is not defined. However, it is evident, by equation (4-25), that the wavenumber content of $W(k,\omega)$ depends on the displacement field outside, as well as inside, the spatial interval $0 \le x \le L$. Inasmuch as our interest is in the wavenumber-frequency description of the displacement field of the string in the interval $0 \le x \le L$, we want to avoid contaminating that description with wavenumber-frequency components arising from any assumed displacement field exterior to this spatial interval. Such contamination is avoided by requiring that $w(x,t) = 0$ for $x < 0$ and $x > L$.

For mathematical convenience, we define the space-time field $w_\infty(x,t)$ as the extension of the mathematical form of equation (4-19) over all x and t. That is,

$$w_\infty(x,t) = \sum_{n=1}^{\infty} \left\{C_n \cos(\omega_n t) + D_n \sin(\omega_n t)\right\} \alpha_n(x) \tag{4-26}$$

for all x and t. By use of equations (4-11) and (4-26), we can then express the desired space-time displacement field as

$$w(x,t) = b(x) w_\infty(x,t) \tag{4-27}$$

for all x and t. It is easily verified that the displacement field defined by equation (4-27) is equivalent to that of equation (4-19) in the spatial interval $0 \le x \le L$ and is zero elsewhere.

By equations (4-25) and (4-27), the wavenumber-frequency transform of the space-time displacement field of the free, finite string with fixed-end conditions can be written in the form

$$W(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(x) w_\infty(x,t) \exp\{-i(kx + \omega t)\}\, dx\, dt \,. \tag{4-28}$$

By (1) substitution of equation (4-26) into equation (4-28), (2) expression of the $\cos(\omega_n t)$ and $\sin(\omega_n t)$ in their exponential forms, and (3) use of

equation (2-38), it is straightforward to show that

$$W(k,\omega) = \pi \sum_{n=1}^{\infty} \{(C_n - iD_n)\delta(\omega - \omega_n) + (C_n + iD_n)\delta(\omega + \omega_n)\} I_n(k) \ , \tag{4-29}$$

where $I_n(k)$ is the spatial Fourier transform of the n-th normal mode, space limited by b(x). That is,

$$I_n(k) = \int_{-\infty}^{\infty} b(x)\alpha_n(x)\exp(-ikx)\ dx \ . \tag{4-30}$$

By expressing the normal mode, defined by equation (4-20), in exponential form, we can write $I_n(k)$ as

$$I_n(k) = \frac{1}{2i}\{B(k - n\pi/L) - B(k + n\pi/L)\} \ , \tag{4-31}$$

where B(k), the Fourier transform of b(x), is easily shown to be

$$B(k) = L\ \exp(-ikL/2)\ \frac{\sin(kL/2)}{(kL/2)} \ . \tag{4-32}$$

Equation (4-29) shows $W(k,\omega)$ for the free, finite string with fixed ends to be a discrete function of ω, with Dirac delta functions at all positive and negative integer multiples of $\pi c_s/L$. At each ω_n, the delta function in ω is multiplied by the wavenumber transform of the corresponding space-limited, n-th normal mode and a weighting factor appropriate to the particular ω_n. The wavenumber transform of the n-th normal mode can be seen, by equations (4-31) and (4-32), to be continuous functions of k for all n. Thus, the wavenumber-frequency transform of the displacement field of the free, finite string with fixed ends is discrete in ω, but continuous in k.

As stated above, the continuous behavior of $W(k,\omega)$ in k, at each ω_n, results from $I_n(k)$, the wavenumber transform of the n-th normal mode, space limited by b(x). By equations (4-31) and (4-32), it is evident that $I_n(k)$

is, in general, a complex quantity and therefore influences both the magnitude and phase of $W(k,\omega)$. Figure 4-1 illustrates the magnitude and phase of $I_6(k)$: that is, $I_n(k)$ for the 6-th normal mode of the fixed-end, finite string.

The magnitude of $I_6(k)$, shown in figure 4-1(a) over the wavenumber range $-12\pi/L \leq k \leq 12\pi/L$, defines the magnitudes of the (complex) amplitudes of the waves of the form $\exp(ikx)$ that comprise the spatial field defined by $b(x)\alpha_6(x)$. By equations (4-11) and (4-20) and use of the exponential form for $\sin(n\pi x/L)$, it can be shown that

$$b(x)\alpha_6(x) = \begin{cases} 0 \ , & -\infty < x < 0 \ , \\ (1/2i)\{\exp(i6\pi x/L) - \exp(-i6\pi x/L)\} \ , & 0 \leq x \leq L \ , \\ 0 \ , & L < x < \infty \ . \end{cases} \qquad (4\text{-}33)$$

By the form of equation (4-33), it is not surprising that the largest contributions to the wavenumber transform of $b(x)\alpha_6(x)$ occur at the wavenumbers $\pm 6\pi/L$. Indeed, it can easily be shown that the Fourier transform of $\alpha_6(x)$ alone, over all space, is the weighted pair of Dirac delta functions, $\delta(k - 6\pi/L)$ and $\delta(k + 6\pi/L)$. It therefore follows that all wavenumber contributions to $I_6(k)$, other than those at $\pm 6\pi/L$, result from the restriction, mathematically imposed by $b(x)$, that the displacement field be equal to zero outside the spatial interval $0 \leq x \leq L$.

Figure 4-1(b) depicts the phase of $I_6(k)$. Physically, this phase can be interpreted as the phase, at $x = 0$, of the various spatial waves of the form $\exp(ikx)$ that comprise the space-limited, 6-th normal mode as a function of the wavenumber (k) characterizing the spatial waveform. The phase is presented modulo 2π in figure 4-1(b), so all discontinuities of magnitude 2π are merely scale adjustments. The phase discontinuities of magnitude π result from the sign changes in $I_6(k)$ associated with the terms of the form $\sin(kL/2)/(kL/2)$ in $B(k - 6\pi/L)$ and $B(k + 6\pi/L)$. These discontinuities occur at all integer multiples of $2\pi/L$, except $\pm 6\pi/L$. In between such discontinuities, the phase decreases linearly (with slope $-L/2$) with increasing k.

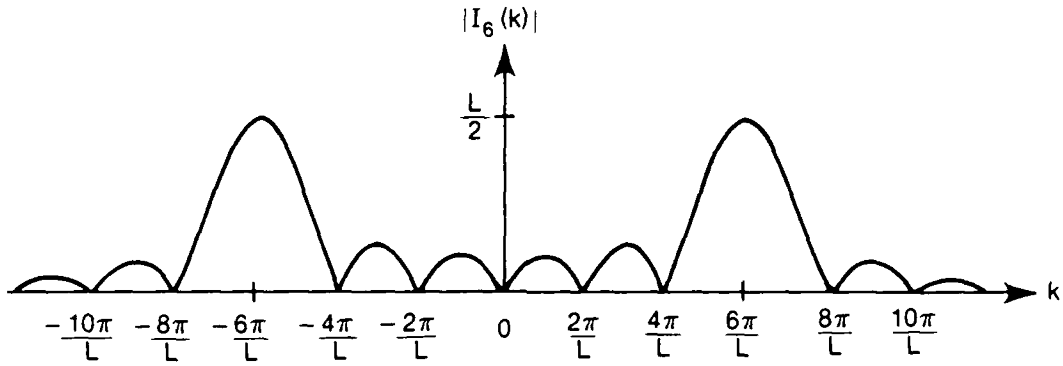

Figure 4-1(a). Magnitude of $I_6(k)$

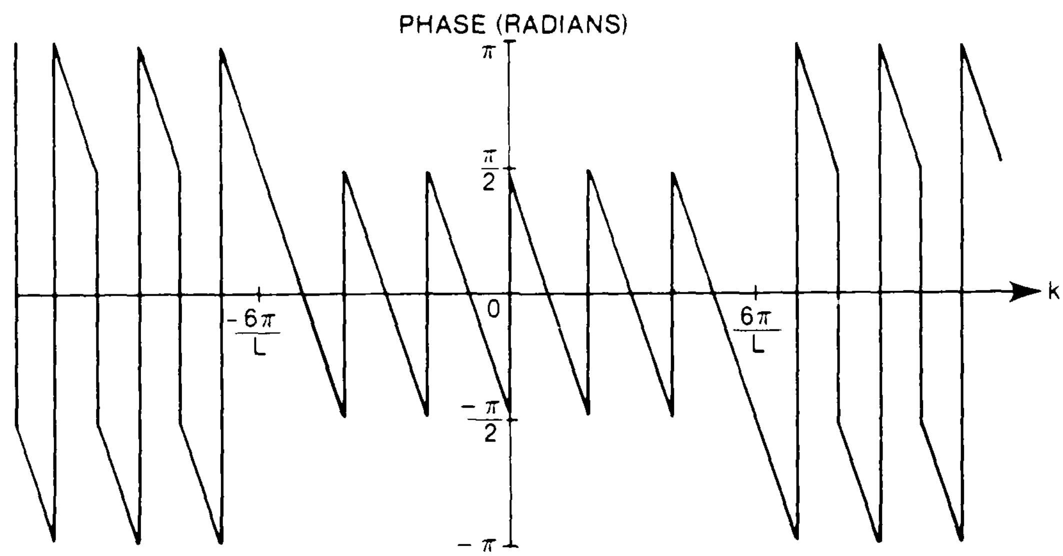

Figure 4-1(b). Phase of $I_6(k)$

Figure 4-1. Magnitude and Phase of $I_n(k)$ for the 6th Mode of the Free, Finite, Fixed-End String

Owing to the discrete nature of $W(k,\omega)$ with frequency, it can be argued, from equation (4-29), that the magnitude of $W(k,\omega)$ can be written as

$$|W(k,\omega)| = \pi \sum_{n=1}^{\infty} \{|C_n - iD_n| \; |I_n(k)| \; \delta(\omega - \omega_n)$$

$$+ |C_n + iD_n| \; |I_n(k)| \; \delta(\omega + \omega_n)\} \; , \qquad (4\text{-}34)$$

where $|\;|$ denotes the absolute value. Inasmuch as C_n and D_n are real constants, it is evident that the wavenumber dependence of $|W(k,\omega)|$ at each modal natural frequency, ω_n, is dictated by $|I_n(k)|$. To illustrate this wavenumber dependence, figure 4-2 presents, in a waterfall-type display, the magnitude of $I_n(k)$ as a function of k at each of the modal natural frequencies in the range $-6\pi c_s/L \leq \omega_n \leq 6\pi c_s/L$. Superposed on this plot are the free wavenumbers, $k = \omega/c_s$, of the infinite, uniform string treated in section 3.3.1 of chapter 3.

By figure 4-2, it is evident that the largest (in magnitude) contribution to $W(k,\omega)$ at each natural frequency (with the exception of $w_{\pm 1}$) occurs at $k \approx \omega_n/c_s$. These contributions are associated with the maxima of $|B(k - n\pi/L) - B(k + n\pi/L)|$, which, by equation (4-31), dictate the wavenumber dependence of $|I_n(k)|$. It should be noted that, although the maximum of $B(k - n\pi/L)$ occurs at $n\pi/L$, the maximum of $|B(k - n\pi/L) - B(k + n\pi/L)|$ is shifted away from $k = n\pi/L$, owing to the interaction between the main lobe of $B(k - n\pi/L)$ and the side lobe of $B(k + n\pi/L)$. The same argument applies to the maximum of $I_n(k)$ at $k = -n\pi/L$. For large values of n, and therefore high modal frequencies, this wavenumber shift is small. However, for the lower order modes, this shift is significant. Indeed, at the first modal frequency (i.e., n = 1), the main lobes of $B(k - \pi/L)$ and $B(k + \pi/L)$ interact to produce a single maximum at k = 0 rather than the expected pair of maxima at $k = \pm\pi/L$.

To the extent that (1) figure 4-2 illustrates, to within a frequency-dependent scale factor, the wavenumber-frequency characteristics of the magnitude of $W(k,\omega)$ associated with the free vibration of a finite,

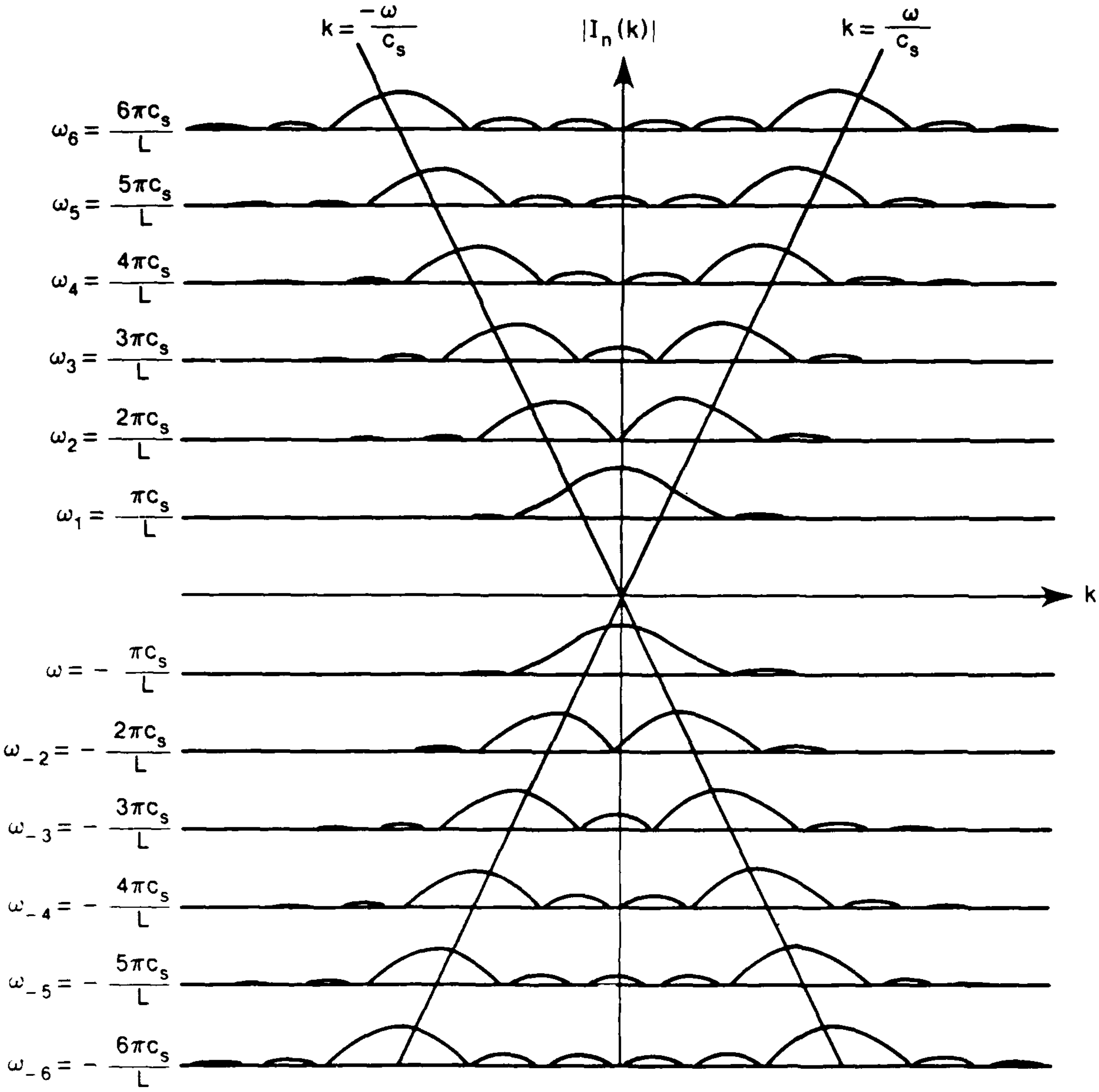

Figure 4-2. $|I_n(k)|$ as a Function of k at Each of the Modal Natural Frequencies in the Range $-6\pi c_s/L \leq \omega_n \leq 6\pi c_s/L$

fixed-end string and (2) only wavenumber-frequency combinations that lie on the lines $k = \pm\omega/c_s$ can contribute to the free vibration of the uniform, infinite string (see chapter 3), figure 4-2 illustrates the two essential differences in the wavenumber-frequency characteristics of the vibration field between the space- and time-invariant string and the space-limited, but time-invariant, string.

The first difference is that the wavenumber-frequency transform of the free vibration field of the infinite (space-invariant) string is a continuous function of frequency along the lines $k = \pm\omega/c_s$, whereas that of the finite (space-limited) string is a discrete function of ω. The reason for this difference can be traced to the boundary conditions. The infinite string, owing to the absence of boundaries, can support the propagation of any wavenumber component introduced by the initial conditions at the frequencies $\omega = \pm kc_s$. The fixed-end, finite string, owing to the boundary condition that $w(0,t) = w(L,t) = 0$, can support propagation, over the spatial interval $0 \le x \le L$, of only that discrete set of wavenumber components associated with the normal modes of the string: that is, by equation (4-20), $k = \pm n\pi/L$. According to the differential equation for the free motion of the finite string (equation (4-15)), the string will support propagation of these discrete wavenumber components only at the corresponding set of discrete frequencies, $\omega = \omega_{\pm n} = \pm n\pi c_s/L$.

The second difference in the wavenumber-frequency transforms of the fields of the finite and infinite, uniform strings is that, at any given frequency, the wavenumber content is discrete for the infinite string, but distributed for the finite string. As discussed above, the string, in free vibration, will only support propagation of waves characterized by $k = \pm\omega/c_s$. As shown in chapter 3, the wavenumber-frequency transform of the displacement field of the infinite string consists, at any frequency, of a weighted pair of Dirac delta functions in k: one at $k = +\omega/c_s$ and one at $k = -\omega/c_s$. It is also evident, by equations (4-19), (4-20), and (4-22), that only the wavenumber frequency combinations related by $k_n = \omega_n/c_s$ (where $k_n = n\pi/L$) satisfy the governing equation for the displacement field of the free, finite string. However, owing to the finite length of the string, we imposed the restriction that the displacement field, $w(x,t)$, be zero outside the physical extent of the string (i.e., $x < 0$ and $x > L$). It was this restriction that introduced, at each natural frequency (ω_n), wavenumber components other than ω_n/c_s in $W(k,\omega)$. Such additional wavenumber components will be present in any space-limited system if the space-time output field of the system is restricted to be zero outside the physical bounds of the system.

Before leaving the example of the free vibrations of the fixed-end, finite string, it is instructive to consider the relationship between the wavenumber

transform of the initial displacement and velocity fields and the modal coefficients C_n and D_n. By writing

$$w_0(x) = 1/(2\pi) \int_{-\infty}^{\infty} W_0(k)\exp(ikx)\, dk \qquad (4\text{-}35)$$

and

$$v_0(x) = 1/(2\pi) \int_{-\infty}^{\infty} V_0(k)\exp(ikx)\, dk \qquad (4\text{-}36)$$

and substituting these expressions into equations (4-23) and (4-24), we can show, by use of equation (4-30), that

$$C_n = 1/(\pi L) \int_{-\infty}^{\infty} W_0(k) I_n^{*}(k)\, dk \qquad (4\text{-}37)$$

and

$$D_n = 1/(n\pi^2 c_s) \int_{-\infty}^{\infty} V_0(k) I_n^{*}(k)\, dk\ , \qquad (4\text{-}38)$$

where the asterisk denotes the complex conjugate. Equations (4-37) and (4-38) show that the modal coefficients, C_n and D_n, are proportional to the integral, over all wavenumbers, of the wavenumber transforms of the respective initial displacement and velocity fields filtered by $I_n^*(k)$, the conjugate of the Fourier transform of the space-limited, n-th normal mode.

Owing to the restriction that w(x,t) is zero outside the spatial interval $0 \leq x \leq L$, neither $W_0(k)$ nor $V_0(k)$ can consist of only a single wavenumber contribution of the form $\delta(k - k_0)$. However, it is interesting to note, by equations (4-37) and (4-38), that each such delta function contribution to W_0 or V_0 produces an infinite number of nonzero modal coefficients, C_n or D_n.

If $W_0(k)$ is proportional to $I_M(k)$, where M is a fixed, positive integer, it is straightforward to show, by equations (4-37), (2-38), and (4-21), that

$$C_n = A\delta_{nM} , \quad (4\text{-}39)$$

where A is the constant of proportionality. Thus, when $W_0(k)$ is proportional to $I_M(k)$, only the modal coefficient C_M contributes to the wavenumber-frequency transform. In a similar fashion, it can be shown that when $V_0(k)$ is proportional to $I_M(k)$, all D_n's, except D_M, are zero. These results stem from the orthogonality of the normal modes over the interval $0 \leq x \leq L$. That is, by the inverse Fourier transformation of equation (4-30),

$$b(x)\alpha_n(x) = 1/(2\pi) \int_{-\infty}^{\infty} I_n(k)\exp(ikx)\, dk . \quad (4\text{-}40)$$

It then follows, by equation (4-21), that the orthogonality condition can be expressed in terms of $I_n(k)$ in the form

$$1/(2\pi) \int_{-\infty}^{\infty} I_m(k) I_n^{*}(k)\, dk = (L/2)\delta_{mn} . \quad (4\text{-}41)$$

The orthogonality condition, as expressed by equation (4-41), can be used to aid the interpretation of the wavenumber filtering of $W_0(k)$ and $V_0(k)$ by $I_n(k)$ in equations (4-37) and (4-38). By use of equations (4-13), (4-14), (4-26), (4-27), and (4-30), it is straightforward to show that

$$W_0(k) = \sum_{n=1}^{\infty} C_n I_n(k) \quad (4\text{-}42)$$

and

$$V_0(k) = \sum_{n=1}^{\infty} \omega_n D_n I_n(k) . \quad (4\text{-}43)$$

By equations (4-42) and (4-43), it is seen that the wavenumber description of the initial conditions can be expressed as a weighted summation of the

wavenumber transforms of the various normal modes. Further, the modal coefficients, C_n and D_n, are the same as those used in the space-time domain to describe w(x,t). Substitution of equations (4-42) and (4-43) into equations (4-37) and (4-38), respectively, yields

$$C_n = 1/(\pi L) \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} C_m I_m(k) I_n^*(k)\, dk \qquad (4\text{-}44)$$

and

$$D_n = 1/(\pi L \omega_n) \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \omega_m D_m I_m(k) I_n^*(k)\, dk \ . \qquad (4\text{-}45)$$

Note that the integrations in equations (4-44) and (4-45) are simply a restatement of the orthogonality condition of equation (4-41). Application of this orthogonality condition to equations (4-44) and (4-45) yields identities for C_n and D_n.

It is clear from the above arguments that the wavenumber description of the initial conditions can be viewed as weighted superpositions of the wavenumber transforms of the space-limited, normal modes of the finite string. The coefficients that weight this superposition are the same coefficients that weight the normal modes in the space-time description of the displacement field, w(x,t).

It should be emphasized that the space-time description of the free vibration field of the fixed-end, finite string given by equation (4-19) applies for all time. In a fashion similar to that observed for the free vibration of the infinite string, the "snapshot" in time of the initial displacement and velocity fields of the finite string determines the modal coefficients C_n and D_n. By equation (4-19), it is evident that, given the normal modes and modal natural frequencies of the finite string, specification of these modal coefficients provides sufficient information to determine the space-time displacement field for all time.

4.2.2 The Finite, Simply Supported Plate

As an example of a space-limited system in two spatial dimensions, $\underline{x} = (x_1, x_2)$, we next treat the free vibrations of the simply supported plate illustrated in figure 4-3. Here, the flexural rigidity, D, and the mass per unit area, μ, of the plate are constant over the physical extent of the plate and are taken to be zero elsewhere. By defining the two-dimensional space-limiting function, $\beta(\underline{x})$, to be

$$\beta(\underline{x}) = \{U(x_1) - U(x_1 - L_1)\}\{U(x_2) - U(x_2 - L_2)\} = \begin{cases} 1\ , & 0 < x_1 < L_1 \text{ and} \\ & 0 < x_2 < L_2\ , \\ 0\ , & \text{otherwise}\ , \end{cases} \tag{4-46}$$

the equation governing the displacement field, $w(\underline{x},t)$, can be written as

$$\beta(\underline{x})\left\{D\nabla^4 w(\underline{x},t) + \mu\,\frac{\partial^2 w(\underline{x},t)}{\partial t^2}\right\} = 0 \tag{4-47}$$

for all $\underline{x}$ and t.

The simply supported boundary conditions require that the displacement and the moment be zero at the boundaries of the plate. Mathematically, the zero displacement condition requires that

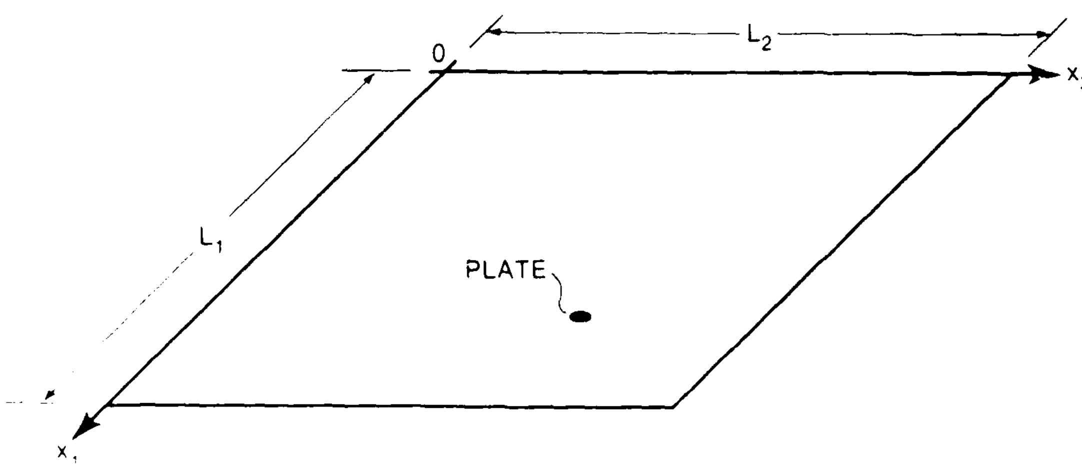

Figure 4-3. Geometry of Simply Supported Plate

$$w(0,x_2,t) = w(L_1,x_2,t) = w(x_1,0,t) = w(x_1,L_2,t) = 0 \tag{4-48}$$

for $0 \leq x_1 \leq L_1$, $0 \leq x_2 \leq L_2$, and all t. The zero-moment boundary conditions can be shown[4] to translate to the mathematical statements

$$\frac{\partial^2 w(0,x_2,t)}{\partial x_1{}^2} = \frac{\partial^2 w(L_1,x_2,t)}{\partial x_1{}^2} = \frac{\partial^2 w(x_1,0,t)}{\partial x_2{}^2} = \frac{\partial^2 w(x_1,L_2,t)}{\partial x_2{}^2} = 0 \tag{4-49}$$

for $0 \leq x_1 \leq L_1$, $0 \leq x_2 \leq L_2$, and all t.

For initial conditions, we again assume that

$$w(\underline{x},0) = w_0(\underline{x}) \tag{4-50}$$

and

$$\frac{\partial w(\underline{x},0)}{\partial t} = v_0(\underline{x}) \; . \tag{4-51}$$

The free vibration of the finite, simply supported plate, like the space-limited string, cannot easily be solved by direct application of Fourier transforms. However, it is easily solved by a separation of variables approach in the space-time domain. The details of this solution procedure are presented in standard texts (see Meirovitch,[5] for example) and will not be reviewed here. The separation of variables solution for the displacement field associated with free vibration of the simply supported plate is

$$w(\underline{x},t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)\} \alpha_{mn}(\underline{x}) \tag{4-52}$$

over $0 \leq x_1 \leq L_1$ and $0 \leq x_2 \leq L_2$ for all t. In equation (4-52), $\alpha_{mn}(\underline{x})$, the normal modes of the plate defined by

$$\alpha_{mn}(\underline{x}) = \sin(m\pi x_1/L_1)\sin(n\pi x_2/L_2) \; , \tag{4-53}$$

form a complete set of orthogonal functions, over the spatial area $0 \leq x_1 \leq L_1$ and $0 \leq x_2 \leq L_2$, that individually satisfy the boundary

conditions of equations (4-48) and (4-49). The orthogonality condition is given by

$$\int_0^{L_1}\int_0^{L_2} \alpha_{mn}(\underline{x})\alpha_{qs}(\underline{x})\, d\underline{x} = (L_1L_2/4)\delta_{mq}\delta_{ns} \ . \qquad (4\text{-}54)$$

The modal natural frequencies of the simply supported plate, denoted by ω_{mn}, are given by

$$\omega_{mn} = (D/\mu)^{1/2}\{(m\pi/L_1)^2 + (n\pi/L_2)^2\} \ . \qquad (4\text{-}55)$$

By use of equations (4-50), (4-51), and (4-52), the orthogonality condition (equation (4-54)) can be used to show that the modal coefficients, A_{mn} and B_{mn}, are related to the initial conditions by

$$A_{mn} = \frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} w_0(\underline{x})\alpha_{mn}(\underline{x})\, d\underline{x} \qquad (4\text{-}56)$$

and

$$B_{mn} = \frac{4}{L_1L_2\omega_{mn}}\int_0^{L_1}\int_0^{L_2} v_0(\underline{x})\alpha_{mn}(\underline{x})\, d\underline{x} \ . \qquad (4\text{-}57)$$

To complete the specification of the displacement field of the free, simply supported plate over all space, we define (as we did for the finite string) the displacement field, $w(\underline{x},t)$, to be zero outside the physical extent of the plate: that is, outside the spatial region $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$. By defining w_∞ to be the extension of the displacement field of equation (4-52) over all $\underline{x}$ (that is,

$$w_\infty(\underline{x},t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \{A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\}\alpha_{mn}(\underline{x}) \qquad (4\text{-}58)$$

for all $\underline{x}$ and $\underline{t}$), we can then use equations (4-46) and (4-58) to express the requisite displacement field over all space and time as

$$w(\underline{x},t) = \beta(\underline{x}) w_{\infty}(\underline{x},t) \ . \tag{4-59}$$

The wavevector-frequency description, $W(\underline{k},\omega)$, of the displacement field is obtained by the following multiple Fourier transform of the space-time field, $w(\underline{x},t)$:

$$W(\underline{k},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(\underline{x},t)\exp\{-i(\underline{k}\cdot\underline{x} + \omega t)\}\ d\underline{x}\ dt \ . \tag{4-60}$$

By substituting equation (4-59) into equation (4-60), it is straightforward to show that

$$W(\underline{k},\omega) = \pi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \{(A_{mn} - iB_{mn})\delta(\omega - \omega_{mn}) + (A_{mn} + iB_{mn})\delta(\omega + \omega_{mn})\} I_{mn}(\underline{k}) \ , \tag{4-61}$$

where

$$I_{mn}(\underline{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta(\underline{x})\alpha_{mn}(\underline{x})\exp(-i\underline{k}\cdot\underline{x})\ d\underline{x} \ . \tag{4-62}$$

By defining the Fourier transform of $\beta(\underline{x})$ as

$$B(\underline{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta(\underline{x})\exp(-i\underline{k}\cdot\underline{x})\ d\underline{x}$$

$$= L_1 L_2 \exp\{-i(k_1 L_1/2 + k_2 L_2/2)\}\left\{\frac{\sin(k_1 L_1/2)}{(k_1 L_1/2)}\right\}\left\{\frac{\sin(k_2 L_2/2)}{(k_2 L_2/2)}\right\} , \tag{4-63}$$

it follows, from equation (4-62), that $I_{mn}(\underline{k})$ can be expressed as

$$I_{mn}(\underline{k}) = (1/4)\{B(k_1 - m\pi/L_1, k_2 + n\pi/L_2) + B(k_1 + m\pi/L_1, k_2 - n\pi/L_2)$$
$$- B(k_1 - m\pi/L_1, k_2 - n\pi/L_2) - B(k_1 + m\pi/L_1, k_2 + n\pi/L_2)\} \ . \tag{4-64}$$

By equations (4-61) through (4-64), it is evident that the wavevector-frequency transform of the displacement field of the simply supported plate is discrete in frequency and continuous in both wavevector components, k_1 and k_2. The discrete frequency components occur as Dirac delta functions at $\pm\omega_{mn}$, where ω_{mn} are the modal natural frequencies of the plate. At each modal natural frequency, the wavevector dependence of $W(\underline{k},\omega)$ is dictated by the product of (1) a complex constant, which depends only on the (real) modal coefficients, and (2) $I_{mn}(\underline{k})$, the wavevector transform of the corresponding space-limited, natural mode of the plate. Note that at $\omega = -\omega_{mn}$, the constant that modifies $I_{mn}(\underline{k})$ is the complex conjugate of the constant that modifies $I_{mn}(\underline{k})$ at $\omega = +\omega_{mn}$.

To aid in the physical interpretation of the wavevector-frequency field given by equation (4-61), it is useful to employ the inverse Fourier transformation of equation (2-56) on equation (4-61) to obtain the following description of the space-time field of the simply supported plate:

$$w(\underline{x},t) = \frac{1}{8\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \{(A_{mn} - iB_{mn}) I_{mn}(\underline{k}) \exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)] + (A_{mn} + iB_{mn}) I_{mn}(\underline{k}) \exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]\} \, d\underline{k} \quad . \tag{4-65}$$

By equation (4-65), it is seen that the space-time displacement field of the simply supported plate is comprised of a superposition of complex harmonic waves of the forms $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ over all wavevectors, $\underline{k}$, and over all discrete natural frequencies, ω_{mn}, of the plate. At each natural frequency, the magnitudes of the complex amplitudes and the initial phases of all harmonic wave components of the form $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ are specified by the product $(A_{mn} - iB_{mn})I_{mn}(\underline{k})$, and the magnitudes and initial phases of harmonic wave components of the form $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ are specified by the product $(A_{mn} + iB_{mn})I_{mn}(\underline{k})$. Note, by equation (4-61), that these are the same products that specify the wavevector dependence of $W(\underline{k},\omega)$ at the discrete frequencies $+\omega_{mn}$ and $-\omega_{mn}$, respectively.

From equation (4-65), it is obvious that the magnitudes of the complex amplitudes of the harmonic waves of the forms $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and

$\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ are equal at any specified wavevector and natural frequency and are given by

$$\frac{|(A_{mn} - iB_{mn})I_{mn}(\underline{k})|}{8\pi^2} = \frac{|(A_{mn} + iB_{mn})I_{mn}(\underline{k})|}{8\pi^2} = \frac{(A_{mn}^2 + B_{mn}^2)^{1/2}\,|I_{mn}(\underline{k})|}{8\pi^2} . \tag{4-66}$$

Note also, from equation (4-61), that because $W(\underline{k},\omega)$ is discrete in ω, the magnitude of $W(\underline{k},\omega)$ is given by

$$|W(\underline{k},\omega)| = \pi \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn}^2 + B_{mn}^2)^{1/2}\,|I_{mn}(\underline{k})|\{\delta(\omega - \omega_{mn}) + \delta(\omega + \omega_{mn})\} . \tag{4-67}$$

By comparison of equations (4-66) and (4-67), it is clear that the wavevector dependence of $|W(\underline{k},\omega)|$ at $\omega = \pm\omega_{mn}$ is, within a factor of $8\pi^3$, equal to the magnitudes of the complex amplitudes of the harmonic wave components $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ that contribute to the space-time displacement field at $\omega = \pm\omega_{mn}$.

Recall that A_{mn} and B_{mn} are constants that depend on the initial displacement and velocity conditions of the plate. It is therefore evident, from equation (4-67), that the wavevector dependence of $|W(\underline{k},\omega)|$ at the discrete frequencies $\omega = \pm\omega_{mn}$ is specified, to within a multiplicative constant, by $|I_{mn}(\underline{k})|$. To illustrate this wavevector dependence, figure 4-4 presents the magnitude of $I_{mn}(\underline{k})$ as a function of k_1 and k_2 for the 6-6th (i.e., m = 6, n = 6) natural mode of a simply supported plate.

By inspection of figure 4-4, it is evident that $|I_{66}(\underline{k})|$ is characterized by four primary maxima located at the wavevectors $(6\pi/L_1, 6\pi/L_2)$, $(6\pi/L_1, -6\pi/L_2)$, $(-6\pi/L_1, 6\pi/L_2)$, and $(-6\pi/L_1, -6\pi/L_2)$. From equation (4-64), it can be established that the amplitudes of these primary maxima are identical and equal to $L_1L_2/4$. Figure 4-4 also shows secondary maxima that occur at odd multiples of π/L_1 along the lines $k_2 = \pm 6\pi/L_2$ and at odd multiples of π/L_2 along the lines $k_1 = \pm 6\pi/L_1$. The amplitudes of

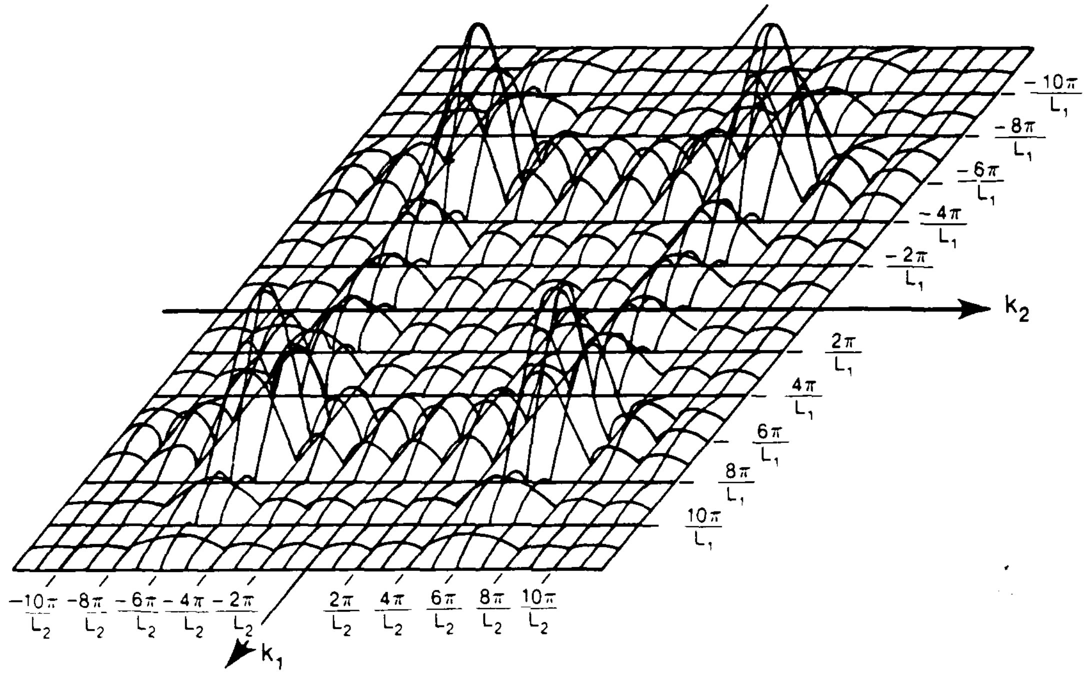

Figure 4-4. Magnitude of $I_{mn}(\underline{k})$ Versus k_1 and k_2 for the 6-6th Mode of a Simply Supported Plate

these secondary maxima can be shown, by equation (4-64), to be less than one-fifth the amplitude of the primary maxima.

It should be emphasized that $|I_{66}(\underline{k})|$ defines the relative magnitude of $|W(\underline{k},\omega)|$ as a function of $\underline{k}$ only at the discrete frequencies $\pm\omega_{66}$. By equation (4-66), $|I_{66}(\underline{k})|$ also defines, as a function of $\underline{k}$, the relative magnitudes of the complex amplitudes of the harmonic waves $\exp[i(\underline{k}\cdot\underline{x} + \omega_{66}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{66}t)]$ that contribute to the space-time displacement field of the plate at the frequencies $\pm\omega_{66}$. By figure 4-4 and the above arguments, it is clear that, although $|I_{66}(\underline{k})|$ is distributed in $\underline{k}$, the wavevector contributions occurring at the wavevectors $(\pm 6\pi/L_1, \pm 6\pi/L_2)$ are significantly larger than those occurring at any other wavevector. Further, the magnitudes of the wavevectors characterizing these primary maxima are equal. If we denote the magnitudes of the wavevectors associated with the primary maxima of $|I_{66}(\underline{k})|$ by k_{66}, it is evident that

$$k_{66} = \{(6\pi/L_1)^2 + (6\pi/L_2)^2\}^{1/2} . \tag{4-68}$$

By use of equations (4-55) and (4-68), it is evident that the magnitudes of the wavevectors associated with the four primary maxima of $|I_{66}(\underline{k})|$ are characterized by

$$k_{66} = \sqrt[4]{\mu\omega_{66}^2/D} \ . \tag{4-69}$$

In chapter 3, the wavenumber characterizing the free vibrations (i.e., the free wavenumber) of a space- and time-invariant plate at any frequency was defined by

$$k_p(\omega) = \sqrt[4]{\mu\omega^2/D} \ . \tag{4-70}$$

By comparison of equations (4-69) and (4-70), it is evident that the magnitudes of the wavevectors associated with the four primary maxima of $|I_{66}(\underline{k})|$ correspond to the free wavenumber of the plate at the natural frequency, ω_{66}, associated with the 6-6th natural mode of the simply supported plate.

The above observations regarding determination of the relative wavevector dependence of $|W(\underline{k},\omega)|$ at the discrete frequencies $\pm\omega_{66}$ by examination of the wavevector characteristics of the magnitude of $I_{66}(\underline{k})$ can be extended to any of the natural frequencies, ω_{mn}. That is, the relative wavevector dependence of $|W(\underline{k},\omega)|$ at ω_{mn} is determined by the magnitude of $I_{mn}(\underline{k})$ for arbitrary values of m and n. Further, from equations (4-63) and (4-64), it is straightforward to show that, for all m and n except unity, $|I_{mn}(\underline{k})|$ is characterized by four primary maxima of amplitude $L_1L_2/4$ occurring at the wavevectors $(m\pi/L_1, n\pi/L_2)$, $(-m\pi/L_1, n\pi/L_2)$, $(m\pi/L_1, -n\pi/L_2)$, and $(-m\pi/L_1, -n\pi/L_2)$. The magnitudes of these four wavevectors are equal and given by

$$k_{mn} = \sqrt[4]{\mu\omega_{mn}^2/D} = k_p(\omega_{mn}) \ . \tag{4-71}$$

For m = 1, the two associated primary maxima at $k_1 = \pm\pi/L_1$ interact, thereby producing a single, broader maximum at $k_1 = 0$. Similar arguments apply in k_2 for n = 1.

The initial phases of the harmonic waves of the form $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ that contribute to the space-time field of the freely vibrating, simply supported plate at each ω_{mn} are seen, from equation (4-65), to be determined by the arguments of the complex products $(A_{mn} - iB_{mn})\, I_{mn}(\underline{k})$ and $(A_{mn} + iB_{mn}) I_{mn}(\underline{k})$, respectively. By equation (4-61), these are the same products that specify the wavevector dependence of $W(\underline{k},\omega)$ at the discrete frequencies $\pm\omega_{mn}$. As the argument of a product is the sum of the arguments of the terms comprising the product, we can gain some insight into the initial phase by examining the individual arguments of $(A_{mn} \pm iB_{mn})$ and $I_{mn}(\underline{k})$.

At any discrete frequency ω_{mn}, the modal coefficients A_{mn} and B_{mn} are real constants. Thus, if we denote the argument of $A_{mn} - iB_{mn}$ by Θ_{mn}, the terms $A_{mn} - iB_{mn}$ and $A_{mn} + iB_{mn}$ contribute constant initial phase shifts of Θ_{mn} and $-\Theta_{mn}$ to the respective harmonic waves $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ that contribute to the space-time field, $w(\underline{x},t)$, at the frequency ω_{mn}.

By equations (4-63) and (4-64), it is clear that the argument of $I_{mn}(\underline{k})$ is a complicated function of $\underline{k}$. However, after some tedious manipulation, it can be shown that

$$\arg\{I_{mn}(\underline{k})\} = -\{k_1L_1/2 + k_2L_2/2 + (m + n)\pi/2\} + j(\underline{k})\pi \,, \qquad (4\text{-}72)$$

where $j(\underline{k})$ is a function that is zero or one, depending on the sign (as a function of $\underline{k}$) of the summation of terms of the form $\{\sin(k_1L_1/2)\sin(k_2L_2/2)\}/\{(k_1L_1/2)(k_2L_2/2)\}$ that arise in $I_{mn}(\underline{k})$ from various combinations of $B(k_1 \pm m\pi/L_1, k_2 \pm n\pi/L_2)$. By equation (4-72), it is seen that, in between the jumps of $\pm\pi$ dictated by $j(\underline{k})$, the argument of $I_{mn}(\underline{k})$ is a linear function of k_1 and k_2. Further, at each discrete frequency, ω_{mn}, the argument has a constant component that depends on $m + n$.

The argument of $I_{mn}(\underline{k})$ described by equation (4-72) is much too complicated to illustrate as a function of k_1 and k_2. However, at ω_{66}, the k_1 dependence of $\arg\{I_{66}(\underline{k})\}$ along the lines $k_2 = \pm 6\pi/L_2$ can be shown to be identical to the argument of $I_6(k)$ for the fixed-end, finite string shown in figure 4-1(b).

By the above arguments, the initial phases, at $\underline{x} = (0,0)$, of the harmonic waves of the form $\exp[i(\underline{k}\cdot\underline{x} + \omega_{mn}t)]$ and $\exp[i(\underline{k}\cdot\underline{x} - \omega_{mn}t)]$ that contribute to $w(\underline{x},t)$ at $\omega = \omega_{mn}$ are given by $\Theta_{mn} + \arg\{I_{mn}(\underline{k})\}$ and $-\Theta_{mn} + \arg\{I_{mn}(\underline{k})\}$, respectively.

By definition of $W_0(\underline{k})$ and $V_0(\underline{k})$ as the wavevector transforms of the initial displacement and velocity fields, $w_0(\underline{x})$ and $v_0(\underline{x})$, respectively, equations (4-56), (4-57), and (4-62) can be used to show that the modal coefficients are related to $W_0(\underline{k})$ and $V_0(\underline{k})$ by

$$A_{mn} = \frac{1}{\pi^2 L_1 L_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_0(\underline{k}) I_{mn}^*(\underline{k})\, d\underline{k} \tag{4-73}$$

and

$$B_{mn} = \frac{1}{\pi^2 L_1 L_2 \omega_{mn}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_0(\underline{k}) I_{mn}^*(\underline{k})\, d\underline{k} \ . \tag{4-74}$$

Equations (4-73) and (4-74) show that the modal coefficients, A_{mn} and B_{mn}, are proportional to respective integrals of the wavevector descriptions of the initial displacement and velocity fields filtered by the conjugate of the wavevector transform of the space-limited, mn-th normal mode.

The restriction that $w(\underline{x},t)$, and thereby $v(\underline{x},t)$, be zero outside the physical extent of the plate precludes $W_0(\underline{k})$ or $V_0(\underline{k})$ from consisting of a single wavevector contribution of the form $\delta(\underline{k} - \underline{k}_0)$, because such a form corresponds to a space-time field of the form $\exp(i\underline{k}\cdot\underline{x})$ over all $\underline{x}$. Rather, by Fourier transformation of equations (4-50), (4-51), and (4-52), over all $\underline{x}$, it is straightforward to show that

$$W_0(\underline{k}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} I_{mn}(\underline{k}) \tag{4-75}$$

and

$$V_0(\underline{k}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} I_{mn}(\underline{k}) \ . \tag{4-76}$$

Thus, it is evident that $W_0(\underline{k})$ and $V_0(\underline{k})$ can be expressed as a weighted superposition of the wavevector transforms of the space-limited, normal modes. Recall that these transforms are continuous in $\underline{k}$. Note also that the weighting coefficients, A_{mn} and B_{mn}, are those used to express $w(\underline{x},t)$ in the space-time domain.

The inverse Fourier transform of equation (4-62) is

$$\beta(\underline{x})\alpha_{mn}(\underline{x}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} I_{mn}(\underline{k}) \exp(i\underline{k}\cdot\underline{x}) \, d\underline{k} \ . \tag{4-77}$$

By equations (4-54) and (4-77), the orthogonality condition on $\alpha_{mn}(x)$ can be translated into the following orthogonality condition on $I_{mn}(k)$:

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} I_{mn}(\underline{k}) I_{qs}^{*}(\underline{k}) \, d\underline{k} = \frac{L_1 L_2}{4} \delta_{mq} \delta_{ns} \ . \tag{4-78}$$

By equation (4-78), it is evident that $I_{mn}(k)$, where m and n are integers between one and infinity, constitute a complete set of orthogonal functions over the interval $(-\infty,-\infty) < \underline{k} < (\infty,\infty)$. This set of functions is the Fourier conjugate of the complete orthogonal set formed by the normal modes, $\alpha_{mn}(\underline{x})$. If equations (4-75) and (4-76) are multiplied by $I_{qs}(\underline{k})$ and integrated over all $\underline{k}$, the orthogonality condition of equation (4-78) can be used to verify the relationships of equations (4-73) and (4-74) between the modal coefficients, A_{mn} and B_{mn}, and the wavevector transforms, $W_0(\underline{k})$ and $V_0(\underline{k})$, of the initial conditions.

4.2.3 Summary of Free Wave Characteristics of Space-Limited Systems

The free responses of the space-limited, time-invariant linear systems treated in the above examples exhibit certain common wavevector frequency

characteristics. This section highlights certain of these common wavevector-frequency characteristics and compares them to the characteristics of the space- and time-invariant systems treated in chapter 3.

The wavevector-frequency description of the free response of space-limited, time-invariant linear systems cannot, in the majority of cases, be easily obtained by direct application of Fourier transforms to the governing equations. Rather, most free, space-limited systems best lend themselves to solution in the space-time domain, where the solution is expressed as a weighted superposition of the normal modes of the system. Each normal mode defines an allowed spatial pattern of free response of the structure. Corresponding to each normal mode is a modal natural frequency, which defines the only frequency at which the system will support the free response defined by the normal mode. The wavevector-frequency description of the response field of a free, space-limited system is obtained by appropriate Fourier transformation of the space-time solution.

The above procedure is in contrast to that employed in chapter 3 to obtain the wavevector-frequency description of the free response of space- and time-invariant linear systems. For these systems, the wavevector-frequency description was obtained by direct transformation of the governing partial differential equations into the wavevector-time domain. The resultant ordinary differential equation in time was solved for the temporal characteristics of the field. Subsequent temporal Fourier transformation of this wavevector-time field led to the desired wavevector-frequency description.

Recall that the wavevector-frequency description (or transform) of the response of a system defines the specific combination of harmonic plane waves that comprise the space-time output field of the system. The space-time field of the free, space-limited system is comprised of a superposition of harmonic waves over an infinite set of discrete frequencies and over all wavevectors at each of the discrete frequencies. The set of discrete frequencies is comprised of all positive and negative values of the (positive) modal natural frequencies of the system. At each discrete frequency, the relative amplitudes and phases of the harmonic waves contributing to the space-time response field are determined by the wavevector transform of the space-limited, normal mode

corresponding to that discrete frequency. Thus, the wavevector-frequency description of the free response of a space-limited system is discrete in frequency, but continuous in the wavevector domain. The continuous nature of the wavevector-frequency description in the wavevector domain at each discrete frequency can be traced to the requirement that the response of the system be zero outside the physical extent of the system.

In contrast, the space-time fields of the free, space- and time invariant systems treated in chapter 3 were comprised of a superposition of harmonic waves over the restricted set of wavevector and frequency combinations that can propagate as free waves in the system. The wavevector-frequency description (or transform) of the response field of a free, space- and time-invariant system is therefore characterized by nonzero values only along those surfaces or lines, in the wavevector-frequency domain, on which the wavevector-frequency combination corresponds to an allowable free wave of the system.

Owing to the absence of external forces in free systems, the wavevectors contributing to the free response of both the space-limited and space-invariant systems are completely determined by those wavevectors present in the initial conditions. In the space-limited system, the wavevector transforms of the initial space-time fields, filtered by the modal wavevector response, determine the modal coefficients and, thereby, the complex amplitudes of the various wavevector components at each modal natural frequency. In the space-invariant system, knowledge of the wavevector content of the initial fields and the wavevector-frequency combinations comprising free waves in the system is sufficient to completely define the wavevector-frequency description of the free response field.

4.3 FORCED RESPONSE OF SPACE-VARYING, TIME-INVARIANT SYSTEMS

The forced response of space-varying, but time-invariant, linear acoustic systems is governed by mathematical models of the form of equation (4-8): that is,

$$_{It}L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t)$$

for all $\vec{x}$ and t. Recall that the space-varying nature of the system is specified by the coefficients $b_{jlmn}(\vec{x})$ in the space-varying, time-invariant linear operator, ${}_{It}L_{\vec{x},t}\{\ \}$, defined by equation (4-7).

In this section, we examine the input-output relationships for space-varying, time-invariant linear systems and present two examples of the forced response of space-limited systems that arise in structural acoustics.

The general input-output relationships for space-varying, time-invariant linear systems are developed from the same basic concepts as those used in chapter 3 to treat space- and time-invariant systems. These concepts are (1) the principle of superposition for linear systems and (2) the Green's function, or impulse response.

The principle of superposition for a general space- and time-varying linear system is described by equations (3-36) through (3-41) in section 3.4.1 of chapter 3. Inasmuch as the space-varying, time-invariant linear systems of interest in this chapter are a subset of space- and time-varying linear systems, the principle of superposition described by these equations applies to space-varying, time-invariant linear systems.

For space varying, time-invariant linear systems, the coefficients, $b_{jlmn}(\vec{x})$, of the linear operator, ${}_{It}L_{\vec{x},t}\{\ \}$, are independent of time, but are functions of the spatial vector, $\vec{x}$. Thus, the form of the linear operator ${}_{It}L_{\vec{x},t}\{\ \}$ is invariant with a change in the temporal origin, but not with a change in the origin of the spatial coordinates. That is, by defining $\tau = t - \theta$, where θ is a constant, equation (4-8) can be written

$${}_{It}L_{\vec{x},\tau+\theta}\{p(\vec{x},\tau + \theta)\} = f(\vec{x},\tau + \theta) \ . \tag{4-79}$$

However, it is easily shown, by equation (4-7), that

$${}_{It}L_{\vec{x},\tau+\theta}\{\ \} = {}_{It}L_{\vec{x},\tau}\{\ \} \ . \tag{4-80}$$

It therefore follows, from equations (4-79) and (4-80), that

$$\mathrm{It}\,L_{\vec{x},\tau}\{p(\vec{x},\tau+\Theta)\} = f(\vec{x},\tau+\Theta) \;. \tag{4-81}$$

We can conclude, by comparison of equations (4-8) and (4-81), that the output of a space-varying, time-invariant linear system resulting from the input $f(\vec{x},\tau+\Theta)$ is $p(\vec{x},\tau+\Theta)$.

It also follows from equation (4-8) that if $\vec{\xi} = \vec{x} - \vec{z}$, where $\vec{z}$ is a constant, then

$$\mathrm{It}\,L_{\vec{\xi}+\vec{z},t}\{p(\vec{\xi}+\vec{z},t)\} = f(\vec{\xi}+\vec{z},t) \;. \tag{4-82}$$

However, it is clear from equation (4-7) that

$$\mathrm{It}\,L_{\vec{\xi}+\vec{z},t}\{\;\} \neq \mathrm{It}\,L_{\vec{\xi},t}\{\;\} \;. \tag{4-83}$$

Therefore, the response of a space-varying linear system to the input $f(\vec{\xi}+\vec{z},t)$ is not equal to $p(\vec{\xi}+\vec{z},t)$.

By applying these concepts of superposition to the Green's function, or space-time impulse response, of space-varying, time-invariant linear systems, we can obtain general expressions for the input-output relations for such systems.

4.3.1 Green's Functions For Space-Varying, Time-Invariant Systems

The complete mathematical description of a system requires not only the specification of a mathematical model governing the system, but specification of the response at the spatial and temporal limits, or boundaries, of the system as well. Recall, from section 4.1, that there are three categories of space-varying systems: (1) uniform, space limited, (2) nonuniform, space limited, and (3) nonuniform, infinite. One primary difference between space-limited and infinite systems is the way the response of these systems is specified at the spatial limits of the system.

For systems of infinite spatial (or temporal) extent, the response of the system at the spatial (or temporal) limits is specified on the basis of

physical realizability, or causality. This requirement for causal response in such systems is satisfied by selecting appropriate combinations of the homogeneous and particular solutions to the partial differential equations governing the behavior of the system. All systems treated in this text are time invariant, and their temporal response characteristics are uniquely determined by causal arguments. For spatially infinite, time-invariant systems, causality conditions are applied to both the spatial and temporal response characteristics to define a unique system response.

In space-limited, time-invariant systems, the response of the system at its spatial limits, or boundaries, is specified as a part of the definition of the system. The response of such space-time systems is then uniquely defined by the governing partial differential equation, augmented by the specification of the required response at the spatial boundaries of the system (and, of course, the conditions for temporal causality). The requirement of a specific response at the spatial boundaries of a system usually implies the existence of external inputs acting on the boundaries of the system. The spatial boundary conditions can therefore be interpreted as an equation, supplementary to that governing the behavior of the system, that defines those external inputs (additional to those applied interior to the boundaries) required to achieve the specified response on the spatial boundaries of the system.

As a consequence of the different forms for specification of the responses of infinite and space-limited systems at their spatial limits, different mathematical procedures are required to formulate the respective input-output relationships for these space-varying systems. It is therefore convenient to treat the Green's function solutions for infinite and space-limited versions of space-varying systems separately.

As a prelude to the development of these Green's function solutions, certain remarks are in order regarding the role and treatment of such solutions in this text.

With regard to the role, the Green's function solution for a linear system relates the output of the system to the input and the Green's function in an integral form. This solution is general in the sense that, given the Green's

function for the linear system of interest, the output field resulting from any input field can be predicted. By appropriate Fourier transformation of the Green's function solution, the wavevector-frequency description (or transform) of the output field can be related to the wavevector-frequency description of the input field and the wavevector-frequency response of the system (i.e., the wavevector-frequency transform of the Green's function). This transformed relationship has the same generality as the Green's function solution. Thus, the role of Green's function solutions in the wavevector-frequency analysis of acoustic systems is to provide the basis from which general input-output relationships can be written in the wavevector-frequency domain for various classes of linear acoustic systems.

Let us now address the treatment of the Green's function in this text. The concept of Green's functions is a simple one. However, any rigorous development of the theory of Green's functions for a general linear space-time system requires mathematically complex and, consequently, lengthy arguments. Morse and Feshbach[6] devote over 100 pages to Green's functions, Courant and Hilbert[7] treat this subject in about 40 pages, and Greenberg[8] devotes an entire book to the development of a consistent theory of Green's functions. While these references vary somewhat in the generality and rigor of their respective treatments of Green's functions, they serve to illustrate the futility of attempting to present a comprehensive treatment of Green's functions in a few pages. Inasmuch as the focus of this book is the wavevector-frequency analysis of acoustic systems, we must conclude that a rigorous treatment of Green's functions is beyond the scope of this text. Consequently, we rely on somewhat heuristic arguments for the development of Green's function solutions of linear systems. For a more thorough treatment of such solutions, the reader is encouraged to consult the references cited above.

With apologies to the reader for this lengthy prelude, we now address the Green's function solution for infinite versions of space-varying linear systems.

4.3.1.1 <u>The Green's Function for Infinite, Nonuniform, Time-Invariant Linear Systems</u>. An infinite, nonuniform, time-invariant linear system is one in which (1) at least one of the coefficients, $b_{jlmn}(\vec{x})$, of the linear

operator ${}_{It}L_{\vec{x},t}\{\ \}$, defined by equation (4-7), varies with $\vec{x}$ and (2) at least one of these coefficients, at any $\vec{x}$, is nonzero. These conditions ensure a continuous system over all space that has space-varying properties. An example of an infinite, nonuniform linear system is an infinitely long, uniformly tensioned string having a mass per unit length that varies (but remains positive) over the length of the string.

Subject to the above restriction on the coefficients $b_{jlmn}(\vec{x})$, the output, $p(\vec{x},t)$, of the infinite, nonuniform, time-invariant linear system resulting from the input, $f(\vec{x},t)$, is governed by equation (4-8). Solutions to this equation are restricted to those that are casual in space and time.

Recall that the Green's function, $g(\vec{x},t;\vec{x}_0,t_0)$, is defined as the response of the system at the spatial position $\vec{x}$ and time t to an impulsive input applied at the spatial location $\vec{x}_0$ and time t_0. Therefore, the Green's function for the infinite, nonuniform, time-invariant linear system is defined by that combination of particular and homogeneous solutions to

$$
{}_{It}L_{\vec{x},t}\{g(\vec{x},t;\vec{x}_0,t_0)\} = \delta(\vec{x} - \vec{x}_0)\delta(t - t_0) \tag{4-84}
$$

that are physically realizable, or causal, over all space and time.

The condition for temporal causality is that the output, or response, cannot anticipate the input in time. Therefore, for the Green's function to be causal, we require that

$$
g(\vec{x},t;\vec{x}_0,t_0) = 0, \quad t < t_0 \ ,
$$

and (4-85)

$$
\frac{\partial^n g(\vec{x},t;\vec{x}_0,t_0)}{\partial t^n} = 0, \quad t < t_0, \quad \text{for all } n.
$$

Spatial causality, for the infinite, nonuniform linear acoustic systems of interest in this text, requires that the Green's function characterizes a response to the impulsive input that either (1) propagates away from the

spatial location of the impulsive input or (2) decays in amplitude with increasing distance from the location of the input.

By noting the temporal form of the input in equation (4-84), we can take advantage of the time invariance of the system, in the form of equation (4-81), to write

$$\mathrm{It}\, L_{\vec{x},t}\{g(\vec{x},\vec{x}_0,t - t_0)\} = \delta(\vec{x} - \vec{x}_0)\delta(t - t_0) \; . \tag{4-86}$$

Thus, by comparison of equations (4-84) and (4-86), the Green's function for the infinite, nonuniform, time-invariant linear system has the mathematical form

$$g(\vec{x},t;\vec{x}_0,t_0) = g(\vec{x},\vec{x}_0,t - t_0) \; . \tag{4-87}$$

Clearly, for this category of space-varying system, the Green's function depends on the two independent variables $\vec{x}$ and $t - t_0$ and on the parameter $\vec{x}_0$.

Let us assume that the causal Green's function defined by cquation (4-86) is known. By use of the sampling property of the Dirac delta function (see equation (2-31)), we may then express the system input, $f(\vec{x},t)$, of equation (4-8) as an integral (i.e., summation) of the product of a weighting function and the delta functions that define the input for the Green's function. That is,

$$f(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{x}_0,t_0)\delta(\vec{x} - \vec{x}_0)\delta(t - t_0)\, d\vec{x}_0\, dt_0 \; . \tag{4-88}$$

By equations (4-87) and (4-88), the principle of superposition for linear systems (see equations (3-39)-(3-41)) can be used to argue that

$$\mathrm{It}\, L_{\vec{x},t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{f(\vec{x}_0,t_0)g(\vec{x},\vec{x}_0,t - t_0)\}\, d\vec{x}_0\, dt_0$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{x}_0,t_0)\delta(\vec{x} - \vec{x}_0)\delta(t - t_0)\, d\vec{x}_0\, dt_0 = f(\vec{x},t) \; . \tag{4-89}$$

By comparison of equations (4-8) and (4-89), it is evident that the causal response of an infinite, nonuniform, time-invariant linear system to an arbitrary input, $f(\vec{x},t)$, is given by

$$p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{x}_0,t_0)g(\vec{x},\vec{x}_0,t - t_0)\, d\vec{x}_0\, dt_0 \ . \qquad (4\text{-}90)$$

By employing the change of temporal variable $\tau = t_0 - t_0$, equation (4-90) may be written in the equivalent form

$$p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{x}_0,t - \tau)g(\vec{x},\vec{x}_0,\tau)\, d\vec{x}_0\, d\tau \ . \qquad (4\text{-}91)$$

Equation (4-90) or (4-91) is the Green's function solution for the infinite, nonuniform type of space-varying, time-invariant linear system.

4.3.1.2 <u>The Green's Function for Space-Limited, Time-Invariant Linear Systems.</u> A space-limited system, as the name implies, is one which exists over some limited portion of space. If, within this limited portion of space, the properties vary with space, the space-limited system is defined to be nonuniform. If the properties are constant over the limited portion of space occupied by the system, the space-limited system is said to be uniform.

Our treatment of Green's function solutions to space-limited systems is based on the approach of Ffowcs-Williams et al.[9] to such systems.

Consider a space-limited system that exists within the volume, V_0, bounded by the surface, S_0. We define the space-limiting function, $s(\vec{x})$, to be

$$s(\vec{x}) = U\{\sigma(\vec{x})\} \ , \qquad (4\text{-}92)$$

where U is the Heaviside function defined by equation (2-32) and $\sigma(\vec{x})$ is a function having the properties

$$\sigma(\vec{x}) > 0 \text{ inside } V_0 ,$$

$$\sigma(\vec{x}) < 0 \text{ outside } V_0 , \tag{4-93}$$

$$\sigma(\vec{x}) = 0 \text{ on } S_0 .$$

Thus, $s(\vec{x})$ defines a function that is one for $\vec{x}$ in V_0 and zero for $\vec{x}$ outside V_0. By use of this space-limiting function, the governing equation for any space-limited system can be written in the form of equation (4-8).

Recall that the system parameters (or properties) are reflected in the space-varying, time-invariant linear operator ${}_{It}L_{\vec{x},t}\{\ \}$ by the coefficients $b_{jlmn}(\vec{x})$. For space-limited systems, such system properties do not pertain outside of the spatial extent, V_0, of the system and can therefore be set to zero for $\vec{x}$ outside V_0. The same argument can be applied to the system input: that is, any input acting outside of V_0 is not acting on the space-limited system and can therefore be set to zero. Thus, for space-limited systems,

$$b_{jlmn}(\vec{x}) = s(\vec{x})\beta_{jlmn}(\vec{x}) \tag{4-94}$$

and

$$f(\vec{x},t) = s(\vec{x})q(\vec{x},t) , \tag{4-95}$$

where $\beta_{jlmn}(\vec{x})$ defines the parameters of the system inside V_0 and $q(\vec{x},t)$ specifies the input to the system inside V_0. Outside V_0, $\beta_{jlmn}(\vec{x})$ and $q(\vec{x},t)$ can be specified arbitrarily. Note that, for a uniform space-limited system, $\beta_{jlmn}(\vec{x})$ is not a function of $\vec{x}$. With $\beta_{jlmn}(\vec{x})$ and $f(\vec{x},t)$ defined by equations (4-94) and (4-95), equation (4-8) describes a space-limited, time-invariant linear system.

We must now address a notational problem. Let us designate the space-varying, time-invariant linear operator (of the form of equation (4-7)), having coefficients $b_{jlmn}(\vec{x})$, by ${}_{It,b}L_{\vec{x},t}\{\ \}$ and an identical operator, having the coefficients $b_{jlmn}(\vec{x})$ replaced by $\beta_{jlmn}(\vec{x})$, by ${}_{It,\beta}L_{\vec{x},t}\{\ \}$.

That is,

$$\mathrm{It},{}_{b}L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J}\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} b_{jlmn}(\vec{x}) \left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} \tag{4-96}$$

and

$$\mathrm{It},{}_{\beta}L_{\vec{x},t}\{\ \} = \sum_{j=0}^{J}\sum_{l=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N} \beta_{jlmn}(\vec{x}) \left(\frac{\partial^j}{\partial x_1{}^j}\right)\left(\frac{\partial^l}{\partial x_2{}^l}\right)\left(\frac{\partial^m}{\partial x_3{}^m}\right)\left(\frac{\partial^n}{\partial t^n}\right)\{\ \} . \tag{4-97}$$

In this notation, the additional presubscript identifies the coefficients of the space-varying linear operator. By use of the notation of equation (4-96), equation (4-8) becomes

$$\mathrm{It},{}_{b}L_{\vec{x},t}\{p(\vec{x},t)\} = f(\vec{x},t) . \tag{4-98}$$

For the forced, space-limited, time-invariant linear system, use of equations (4-94), (4-95), and (4-97) allows us to rewrite equation (4-98) in the mathematically equivalent form

$$s(\vec{x})\,{}_{\mathrm{It},\beta}L_{\vec{x},t}\{p(\vec{x},t)\} = s(\vec{x})q(\vec{x},t) . \tag{4-99}$$

Equation (4-99) is the typical form of the governing equation for forced, space-limited, time-invariant linear systems. To complete the specification of the space-limited problem, the output, $p(\vec{x},t)$, is subject to certain restrictions (i.e., boundary conditions) on the boundary S_0.

Inasmuch as the system is space-limited to within V_0, the desired output of the system is also space limited and is nonzero only within V_0 and on S_0. This desired output can be achieved by transferring the space-limiting function, $s(\vec{x})$, inside the linear operator. For the terms in the linear operator containing no spatial derivatives, this transfer presents no problem. For example,

$$s(\vec{x})\beta_{0001}(\vec{x}) \frac{\partial p(\vec{x},t)}{\partial t} = \beta_{0001}(\vec{x}) \frac{\partial\{s(\vec{x})p(\vec{x},t)\}}{\partial t} . \tag{4-100}$$

However, in the transfer of $s(\vec{x})$ inside the linear operator, each spatial derivative generates an additional term. For example, it is easily verified, by use of equations (4-92) and (2-32), that

$$s(\vec{x})\beta_{1000}(\vec{x}) \frac{\partial p(\vec{x},t)}{\partial x_1} = \beta_{1000}(\vec{x})\frac{\partial\{s(\vec{x})p(\vec{x},t)\}}{\partial x_1}$$

$$- \beta_{1000}(\vec{x})p(\vec{x},t)\delta\{\sigma(\vec{x})\} \frac{\partial\sigma(x)}{\partial x_1} . \qquad (4\text{-}101)$$

It follows that higher spatial derivatives will generate not only terms involving products of $p(\vec{x},t)\delta\{\sigma(\vec{x})\}$, but also additional terms involving products of various order spatial derivatives of $p(\vec{x},t)$ and $\delta\{\sigma(\vec{x})\}$.

The above arguments demonstrate that transferring $s(\vec{x})$ inside the linear operator gives rise to additional terms involving products of $p(\vec{x})$ or its spatial derivatives with $\delta\{\sigma(\vec{x})\}$ or its derivatives. Inasmuch as the Dirac delta function and its derivatives are zero everywhere except at the zeros of the argument of the delta function, these additional terms can be interpreted as additional inputs concentrated at those spatial locations where $\sigma(\vec{x}) = 0$. However, by equation (4-93), these locations are on the bounding surface of the system. Thus, the additional terms correspond to inputs, additional to $q(\vec{x},t)$, that act on the boundary, S_0, of the system.

If we denote the collection of these products of $p(\vec{x})$ or its spatial derivatives with $\delta\{\sigma(\vec{x})\}$ or its derivatives by $\Sigma Q\{p(\vec{x},t),\delta(\sigma)\}$, it can be shown, by arguments similar to those of equation (4-101), that equation (4-99) can be written in the form

$$_{It,\beta}L_{\vec{x},t}\{s(\vec{x})p(\vec{x},t)\} = s(\vec{x})q(\vec{x},t) + \Sigma Q\{p(\vec{x},t),\delta(\sigma)\} \qquad (4\text{-}102)$$

for all $\vec{x}$ and t.

Equation (4-102) is the governing equation for the generalized function $s(\vec{x})p(\vec{x},t)$ that is valid for all space and time. The field described by this generalized function is equal to $p(\vec{x},t)$ in the volume of interest, V_0, and is zero elsewhere. The boundary conditions, in terms of appropriate

specification of $p(\vec{x},t)$ and its spatial derivatives on S_0, are the weighting functions of the additional inputs described in $\Sigma Q\{p(\vec{x},t),\delta(\sigma)\}$. Thus, by absorbing the space-limiting function inside the linear operator, we have transformed a finite space problem with boundary conditions to an infinite space problem with additional inputs concentrated on the boundary.

The Green's function for the space-limited, time-invariant linear system, as defined by equation (4-102), must satisfy

$$I_{t,\beta}L_{\vec{x},t}\{g(\vec{x},\vec{x}_0,t - t_0)\} = \delta(\vec{x} - \vec{x}_0)\delta(t - t_0) \tag{4-103}$$

over all $\vec{x}$ and t for $\vec{x}_0$ in V_0 or on S_0. For $\vec{x}_0$ outside V_0 and S_0, the right-hand side can be replaced by any distribution of sources. The Green's function must satisfy temporal causality and must also satisfy some appropriate number of spatial constraints or conditions in order that it be a unique solution to equation (4-103). For the moment, we leave these spatial conditions unspecified.

We note that the input to equation (4-102) can be expressed as a weighted superposition of the inputs to equation (4-103). That is,

$$s(\vec{x})q(\vec{x},t) + \Sigma Q\{p(\vec{x},t),\delta[\sigma(\vec{x})]\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\vec{x} - \vec{x}_0)\delta(t - t_0)[s(\vec{x}_0)q(\vec{x}_0,t_0) + \Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\, d\vec{x}_0\, dt_0 . \tag{4-104}$$

By assuming that a temporally causal form of the Green's function is known and by once again employing the principle of superposition for linear systems (see section 3.4.1), it follows that

$$I_{t,\beta}L_{\vec{x},t}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[s(\vec{x}_0)q(\vec{x}_0,t_0) + \Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\, d\vec{x}_0\, dt_0\right\}$$

$$= s(\vec{x})q(\vec{x},t) + \Sigma Q\{p(\vec{x},t),\delta[\sigma(\vec{x})]\} . \tag{4-105}$$

By comparison of equations (4-102) and (4-105), it is evident that

$$s(\vec{x})p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[s(\vec{x}_0)q(\vec{x}_0,t_0)$$

$$+ \Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\, d\vec{x}_0\, dt_0 . \qquad (4\text{-}106)$$

Because we assumed a temporally causal Green's function, the output field $s(\vec{x})p(\vec{x},t)$ also satisfies temporal causality. However, we have not yet identified the spatial conditions, or constraints, used to uniquely specify the Green's function. The fact of the matter is that equation (4-106) is a valid representation of the space-limited output for any set of spatial constraints sufficient to provide a unique specification of the Green's function. That is not to say, however, that one can obtain a solution to equation (4-106) for the space-limited output field for an arbitrary choice of spatial constraints on the Green's function. Rather, in applying the Green's function approach to space-limited systems, there is an element of art in specifying the spatial constraints on the Green's function.

Note that the first term in the integrand of equation (4-106) is simply the contribution to the output from the inputs within V_0. The second term in the integrand represents the contributions to the output from the additional inputs on the bounding surface, S_0, of the system. The goal, in selecting the spatial constraints that uniquely specify the Green's function, is to obtain the simplest, solvable mathematical form of equation (4-106). For systems with inputs in V_0, it is desirable to specify the spatial constraints such that the terms related to the additional surface inputs vanish. Such a choice leads to what Ffowcs-Williams[9] calls the "exact Green's function." If inputs are applied only to the system boundaries (i.e., $q(\vec{x}) = 0$ in V_0), then it is desirable to choose spatial constraints that minimize the mathematical complexity of the integral containing the surface inputs. The art of specifying such spatial constraints for space-limited systems can best be illustrated by an example.

Consider the acoustic pressure $p(\vec{x},t)$ in the semi-infinite space $x_3 \geq 0$ resulting from source type inputs, $q(\vec{x},t)$, in the space $x_3 \geq 0$. At the boundary $x_3 = 0$, either $p(\underline{x},0,t)$ or $\partial p(\underline{x},0,t)/\partial x_3$, where $\underline{x} = (x_1,x_2)$, is specified.

In the form of equation (4-99), the governing equation can be written

$$U(x_3)\left\{\nabla^2 p(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2 p(\vec{x},t)}{\partial t^2}\right\} = -U(x_3)q(\vec{x},t) \ , \tag{4-107}$$

where U( ) denotes the Heaviside function. It is easily verified, by use of equation (2-32), that

$$\nabla^2\{U(x_3)p(\vec{x},t)\} = U(x_3)\nabla^2 p(\vec{x},t) + \delta(x_3)\frac{\partial p(\vec{x},t)}{\partial x_3} + \frac{\partial}{\partial x_3}\{p(\vec{x},t)\delta(x_3)\} \ . \tag{4-108}$$

Thus, equation (4-107) can be rewritten in the form of equation (4-102) as

$$\nabla^2\{U(x_3)p(\vec{x},t)\} - \frac{1}{c^2}\frac{\partial^2\{U(x_3)p(\vec{x},t)\}}{\partial t^2}$$

$$= -U(x_3)q(\vec{x},t) + \delta(x_3)\frac{\partial p(\vec{x},t)}{\partial x_3} + \frac{\partial}{\partial x_3}\{p(\vec{x},t)\delta(x_3)\} \ . \tag{4-109}$$

The Green's function is defined as the solution to

$$\nabla^2 g(\vec{x},\vec{x}_0,t - t_0) - \frac{1}{c^2}\frac{\partial^2 g(\vec{x},\vec{x}_0,t - t_0)}{\partial t^2} = -\delta(\vec{x} - \vec{x}_0)\delta(t - t_0) \tag{4-110}$$

that satisfies temporal and spatial causality. It then follows, by equations (4-104), (4-106), (4-109), and (4-110), that

$$U(x_3)p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)\left\{U(x_{30})q(\vec{x}_0,t_0) - \delta(x_{30})\frac{\partial p(\vec{x}_0,t_0)}{\partial x_{30}}\right.$$

$$\left. - \frac{\partial}{\partial x_{30}}[p(\vec{x}_0,t_0)\delta(x_{30})]\right\} d\vec{x}_0 \, dt_0 \ . \tag{4-111}$$

where $\vec{x}_0 = [x_{10}, x_{20}, x_{30}]$.

By integrating the terms containing $\delta(x_{30})$ on x_{30} (the first term directly and the second term by parts), we obtain

$$U(x_3)p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)U(x_{30})q(\vec{x}_0,t_0)\, d\vec{x}_0\, dt_0$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{p(\underline{x}_0,0,t_0)\,\frac{\partial g(\vec{x};\underline{x}_0,0;t - t_0)}{\partial x_{30}} - g(\vec{x};\underline{x}_0,0;t - t_0)\,\frac{\partial p(\underline{x}_0,0,t_0)}{\partial x_{30}}\right\} d\underline{x}_0\, dt_0\ , \qquad (4\text{-}112)$$

where $\underline{x}_0$ denotes $[x_{10}, x_{20}]$. Note, by equation (4-112), that the space-limited field $U(x_3)p(\vec{x},t)$ is expressed as the sum of a volume integral and a surface integral. The volume integral represents the contribution to the field from all inputs within the space $x_3 > 0$. The surface integral represents the contributions associated with those inputs on the surface $x_3 = 0$ required to produce the desired boundary conditions.

For specific types of inputs and boundary conditions in this acoustic half space, we can use equation (4-112) to illustrate the rationale for selecting spatial constraints on the Green's function.

Consider first the case where the input, $q(\vec{x},t)$, is nonzero. If the pressure field at the boundary $x_3 = 0$ is specified to be $p(\underline{x},0,t) = 0$ for all $\underline{x}$ and t, it is immediately evident, by equation (4-112), that if we subject the Green's function to the spatial restriction $g(\vec{x};\underline{x}_0,0;t - t_0) = 0$ for all $\vec{x}$, $\underline{x}_0$, and $t - t_0$, then the surface integral vanishes and the space-limited pressure field is given by

$$U(x_3)p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)U(x_{30})q(\vec{x}_0,t_0)\, d\vec{x}_0\, dt_0\ . \qquad (4\text{-}113)$$

Because the spatial restriction on the Green's function causes the surface integral to vanish, this Green's function is, by definition, exact. Spatial restrictions leading to an exact Green's function can also be defined when $\partial p(\vec{x},t)/\partial x_3 = 0$ at the boundary $x_3 = 0$ for all $\underline{x}$ and t. That is, by equation (4-112), it is evident that if we apply the restriction $\partial g(\vec{x};\underline{x}_0,0; t - t_0)/ \partial x_{30} = 0$ to the Green's function for all $\vec{x}$, $\underline{x}_0$, and $t - t_0$, then the surface integral vanishes and the space-limited pressure field is given in the form of equation (4-113).

Consider now the case where the input, $q(\vec{x},t)$, is equal to zero for all $\vec{x}$ and t. If either $p(\vec{x},t)$ or $\partial p(\vec{x})\partial x_3$ is specified to be zero on the boundary $x_3 = 0$, then it follows, from equation (4-113) and the uniqueness of $U(x_3)p(\vec{x},t)$, that the output pressure field is zero for all space and time. Thus, when $q(\vec{x},t) = 0$, the system has a nonzero response only if $p(\vec{x},t)$ or $\partial p(\vec{x},t)/\partial x_3$ is specified to be nonzero at $x_3 = 0$. If $p(\vec{x},t)$ is specified to be $p_0(\underline{x},t)$ at $x_3 = 0$, it follows from equation (4-112) that the simplest mathematical expression for the space-limited output results when we require that $g(\vec{x};\underline{x},0;t - t_0) = 0$ for all $\vec{x}$, $\underline{x}_0$, and $t - t_0$. In this case, the space-limited output field is related to the specified pressure at the boundary by

$$U(x_3)p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(\underline{x}_0,t_0) \frac{\partial g(\vec{x};\underline{x}_0,0;t - t_0)}{\partial x_{30}} \, d\underline{x}_0 \, dt_0 \, . \qquad (4\text{-}114)$$

By similar arguments, a spatial constraint can be applied to the Green's function to reduce equation (4-112) to a single integral when $\partial p/\partial x_3$ is specified to be nonzero at $x_3 = 0$. The specification of this constraint is left as an exercise for the reader.

The above examples illustrate the manner by which spatial constraints on the Green's function can be selected to simplify the mathematical form of the solution for one particular type of space-limited system. It should be emphasized that the results of these examples cannot be extended to other systems because the additional inputs generated at the boundaries by incorporating the space-limiting function inside the linear operator depend on the form of the linear operator governing the system.

For an arbitrary, space-limited, time-invariant linear system, the general Green's function solution is given by equation (4-106). However, without knowledge of (1) the form of the governing partial differential equation, (2) the definition of the space-limiting function, and (3) the boundary conditions for the particular system of interest, the specific mathematical form for the additional inputs on the bounding surfaces cannot be defined.

4.3.2 The Wavevector-Frequency Response of Space-Varying Systems

By appropriate Fourier transformations of the Green's function solutions for the forced response of space-varying systems, the wavevector-frequency transform of the output field can be related to the corresponding transform of the input field.

Owing to the differences in the mathematical forms of the Green's function solutions between the infinite, nonuniform and the space-limited types of space-varying systems, it is convenient to treat the wavevector-frequency responses of these two types of systems separately. We will start with the infinite, nonuniform type.

4.3.2.1 Wavevector-Frequency Response of Infinite, Nonuniform, Time-Invariant Linear Systems. The space-time output field for an infinite, nonuniform, time-invariant linear system is related to the space-time input field and the Green's function by equation (4-91). We first express the space-time input field as the superposition of harmonic plane waves in the form of equation (2-47). That is,

$$f(\vec{x},t) = (2\pi)^{-4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\vec{\alpha},\Omega)\exp\{i(\vec{\alpha}\cdot\vec{x} + \Omega t)\}\, d\vec{\alpha}\, d\Omega \ , \tag{4-115}$$

where $\vec{\alpha}$ and Ω denote, respectively, the wavevector and frequency components of the input field. Substitution of equation (4-115) into equation (4-91) yields

$$p(\vec{x},t) = (2\pi)^{-4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,\tau)F(\vec{\alpha},\Omega) \exp\{i[\vec{\alpha}\cdot\vec{x}_0 + \Omega(t - \tau)]\}\, d\vec{x}_0\, d\tau\, d\vec{\alpha}\, d\omega \ . \tag{4-116}$$

If we define the wavevector-frequency transform of the output field by

$$P(\vec{k},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\vec{x},t)\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\}\, d\vec{x}\, dt \;, \qquad (4\text{-}117)$$

it follows, from equations (4-116) and (2-38), that

$$P(\vec{k},\omega) = (2\pi)^{-3} \int_{-\infty}^{\infty} G(\vec{k},-\vec{\alpha},\omega)F(\vec{\alpha},\omega)\, d\vec{\alpha} \;, \qquad (4\text{-}118)$$

where $-\vec{\alpha}$ denotes the vector $(-\alpha_1,-\alpha_2,-\alpha_3)$ and $G(\vec{k},\vec{\alpha},\omega)$ is the two-wavevector-frequency response of the system, defined by

$$G(\vec{k},\vec{\alpha},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,\tau)\exp\{-i(\vec{k}\cdot\vec{x} + \vec{\alpha}\cdot\vec{x}_0 + \omega\tau)\}\, d\vec{x}\, d\vec{x}_0\, d\tau \;. \qquad (4\text{-}119)$$

Equation (4-118) relates the wavevector-frequency description, or transform, of the output of an infinite, nonuniform, time-invariant linear system to the corresponding description of the input and the two-wavevector-frequency response of the system. Note that for this space-varying system, the wavevector-frequency transform of the output field is expressed in terms of an integral of the wavevector-frequency transform of the input field and the two-wavevector-frequency response of the system. This is in contrast to the algebraic relationship obtained in the wavevector-frequency domain for the space- and time-invariant systems (see section 3.4.3). Further, for the infinite, nonuniform system, the wavevector-frequency response of the system is seen to be a function of two wavevector variables, whereas the wavevector-frequency response of the space-invariant system was a function of a single wavevector. This, of course, is a consequence of the separate dependence of the Green's function for the space-varying system on $\vec{x}$ and $\vec{x}_0$, whereas the Green's function for the space-invariant system depended only on the difference between $\vec{x}$ and $\vec{x}_0$.

To obtain a physical interpretation of the two-wavevector-frequency response of the infinite, nonuniform, space varying system, consider the output

field resulting from the single harmonic plane wave input field described by

$$f(\vec{x},t) = \exp\{i(\vec{k}_0 \cdot \vec{x} + \omega_0 t)\} \ . \tag{4-120}$$

The wavevector-frequency transform of the input field is then given by

$$F(\vec{k},\omega) = (2\pi)^4 \delta(\vec{k} - \vec{k}_0)\delta(\omega - \omega_0) \ , \tag{4-121}$$

so, by equation (4-118),

$$P(\vec{k},\omega) = 2\pi G(\vec{k},-\vec{k}_0,\omega_0)\delta(\omega - \omega_0) \ . \tag{4-122}$$

However, inasmuch as

$$p(\vec{x},t) = (2\pi)^{-4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\vec{k},\omega)\exp\{i(\vec{k}\cdot\vec{x} + \omega t)\} \, d\vec{k} \, d\omega \ , \tag{4-123}$$

$P(\vec{k},\omega)$ represents the amplitudes and initial phases of the various harmonic plane waves comprising the output field. It therefore follows, by equations (4-120)-(4-123), that $2\pi G(\vec{k},-\vec{k}_0,\omega_0)$ represents the amplitude and initial phase of the harmonic plane wave component of the output field characterized by the wavevector $\vec{k}$ and frequency ω_0 resulting from excitation of the space-varying system by the harmonic plane wave characterized by the wavevector $\vec{k}_0$ and the frequency ω_0. Thus, for infinite, nonuniform, space-varying systems, the two-wavevector-frequency response, $G(\vec{k},\vec{\alpha},\omega)$, defines, at each frequency, the response of the system at the wavevector $\vec{k}$ resulting from excitation of the system at the wavevector $-\vec{\alpha}$.

The conversion, by the infinite, nonuniform, space-varying system, of one wavevector component of the input field into different wavevector components of the output field is called wavevector conversion. Recall that, in space- and time-invariant systems (see equation (3-59)), each wavevector component of the input produces, at any frequency, only the corresponding wavevector component in the output field. Therefore, wavevector conversion does not occur in space-invariant systems. We can therefore conclude that wavevector

conversion is a characteristic of space-varying systems that results from the space-varying properties of the system.

Given knowledge of the wavevector-frequency transform of the input field and the two-wavevector-frequency response of an infinite, nonuniform system, one can (in theory) predict the wavevector-frequency transform of the output field from equation (4-118). Further, as illustrated by equations (4-120)-(4-122), one can determine the two-wavevector-frequency response of the system, $G(\vec{k},\vec{\alpha},\omega)$, as a function of $\vec{k}$ at any desired wavevector, $\vec{\alpha}$, and frequency, ω, by exciting the system by a single plane harmonic wave characterized by the wavevector $-\vec{\alpha}$ and frequency ω and observing the output as a function of $\vec{k}$. However, it is evident from equation (4-118) that, given knowledge of the wavevector-frequency transform of the output field and the two-wavevector-frequency response of the system, one is faced with the solution of an integral equation to determine the wavevector-frequency transform of the input field.

4.3.2.2. Wavevector-Frequency Response of Space-Limited, Time-Invariant Linear Systems. The space-time output field of a space-limited system is related to the space-time input field, the boundary conditions, and the Green's function by equation (4-106). For brevity, let us designate the space-limited input and output fields by

$$f(\vec{x},t) = s(\vec{x})q(\vec{x},t) \tag{4-124}$$

and

$$o(\vec{x},t) = s(\vec{x})p(\vec{x},t) \ , \tag{4-125}$$

respectively. By substituting equations (4-124) and (4-125) into equation (4-106), we obtain

$$o(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[f(\vec{x}_0,t_0) + \Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\, d\vec{x}_0\, dt_0 \ . \tag{4-126}$$

We will deal with the wavevector-frequency response of space-limited systems by considering three special cases of equation (4-126).

We first consider the case in which the boundary conditions imposed on the system are such that it is possible to define an exact Green's function. Recall that an exact Green's function is one defined in such a fashion that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[\Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\, d\vec{x}_0\, dt_0 = 0 \,, \qquad (4\text{-}127)$$

and therefore the output of the space-limited system is given by

$$o(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)f(\vec{x}_0,t_0)\, d\vec{x}_0\, dt_0 \,. \qquad (4\text{-}128)$$

By following the same arguments used for the infinite, nonuniform space-varying system, it is straightforward to show that the wavevector-frequency transform of the space-limited output field, $O(\vec{k},\omega)$, is related to the wavevector-frequency transform of the space-limited input field, $F(\vec{k},\omega)$, by

$$O(\vec{k},\omega) = (2\pi)^{-3}\int_{-\infty}^{\infty} G(\vec{k},-\vec{\alpha},\omega)F(\vec{\alpha},\omega)\, d\vec{\alpha} \,, \qquad (4\text{-}129)$$

where $G(\vec{k},\vec{\alpha},\omega)$ is the two-wavevector-frequency response of the space-limited system and is mathematically defined by equation (4-119).

The form of equation (4-129) is exactly the same as equation (4-118), and the interpretations of this result and of the two-wavevector-frequency response of the space-limited system are identical to those given for the infinite, nonuniform system.

For the second case, consider a space-limited system with boundary conditions and Green's function specified such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[\Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\; d\vec{x}_0\; dt_0 \neq 0\;, \tag{4-130}$$

but such that the product of the Green's function and the additional forces imposed by the boundary constraints is known. Separate knowledge of this product and the Green's function is equivalent to knowledge of $\Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}$. Thus, for this case, we assume that the distribution of inputs at the boundary is known and is designated by $f_s(\vec{x},t)$. That is,

$$f_s(\vec{x},t) = \Sigma Q\{p(\vec{x},t),\delta[\sigma(\vec{x})]\}\;. \tag{4-131}$$

By using the notation of equations (4-124), (4-125), and (4-131) for the space-limited inputs within the boundaries, the space-limited output, and the distribution of inputs on the boundary, respectively, equation (4-126) can be rewritten as

$$o(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[f(\vec{x}_0,t_0) + f_s(\vec{x}_0,t_0)]\; d\vec{x}_0\; dt_0\;. \tag{4-132}$$

It follows, by arguments similar to those used above, that the wavevector-frequency transform of the space-limited output is given by

$$O(\vec{k},\omega) = (2\pi)^{-3}\int_{-\infty}^{\infty} G(\vec{k},-\vec{\alpha},\omega)\{F(\vec{\alpha},\omega) + F_s(\vec{\alpha},\omega)\}\; d\vec{\alpha}\;, \tag{4-133}$$

where $F_s(\vec{\alpha},\omega)$ denotes the wavevector-frequency transform $f_s(\vec{x},t)$. Note that, with the exception of the presence of the additional input term associated with the boundary forces, the form of equation (4-133) is identical to that of equation (4-129).

As an example of a space-limited system with known boundary inputs, consider the semi-infinite acoustic system described in section 4.3.1.2, where

the source inputs, $q(\vec{x},t)$, in the space $x_3 > 0$ and the normal derivative of the pressure field at $x_3 = 0$ are specified. The general Green's function solution for this problem is given by equation (4-112). For notational simplicity, we define

$$o(\vec{x},t) = U(x_3)p(\vec{x},t)$$

and (4-134)

$$f(\vec{x},t) = U(x_3)q(\vec{x},t) .$$

Also, we specify the normal derivative of the pressure at $x_3 = 0$ to be

$$\frac{\partial p(\underline{x},0,t)}{\partial x_3} = a(\underline{x},t) , \tag{4-135}$$

where $a(\underline{x},t)$ is a known function of $\underline{x}$ and t.

For this example, the Green's function is uniquely specified by requiring that the solution to equation (4-110) be restricted by

$$\frac{\partial g(\vec{x};\underline{x}_0,0;t - t_0)}{\partial x_{30}} = 0 . \tag{4-136}$$

Thus, by use of equations (4-112) and (4-134)-(4-136), the space-limited pressure field is given by

$$o(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)f(\vec{x}_0,t_0)\, d\vec{x}_0\, dt_0$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x};\underline{x}_0,0;t - t_0)a(\underline{x}_0,t_0)\, d\underline{x}_0\, dt_0 . \tag{4-137}$$

By use of the inverse of equation (4-119), that is,

$$g(\vec{x},\vec{x}_0,\tau) = (2\pi)^{-7} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{k},\vec{\alpha},\omega)\exp\{i(\vec{k}\cdot\vec{x} + \vec{\alpha}\cdot\vec{x}_0 + \omega\tau)\}\ d\vec{k}\ d\vec{\alpha}\ d\omega\ , \tag{4-138}$$

the wavevector-frequency transform of the space-limited pressure field, $O(\vec{k},\omega)$, can be related to the wavevector-frequency transforms of the space-limited source distribution and the specified pressure gradient on the boundary, $F(\vec{k},\omega)$ and $A(\underline{k},\omega)$, respectively, by

$$O(\vec{k},\omega) = (2\pi)^{-3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\vec{k},-\vec{\alpha},\omega)\{F(\vec{\alpha},\omega) - A(\underline{\alpha},\omega)\}\ d\vec{\alpha}\ . \tag{4-139}$$

Clearly, equation (4-139) has the mathematical form of equation (4-133) with $F_s(\vec{\alpha},\omega)$ independent of α_3.

As the final case of space-limited systems, consider a system with boundary conditions and Green's function specified such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)[\Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}]\ d\vec{x}_0\ dt_0 \neq 0\ , \tag{4-140}$$

but such that some terms resulting from the product of the Green's function and the additional inputs associated with the boundary constraints are not known. This situation can arise when, regardless of the restrictions imposed on the Green's function at the spatial limits of the system, the specified boundary conditions do not provide the information required for the integrand of equation (4-140) to be completely known.

In this case, the Green's function $g(\vec{x},\vec{x}_0,t - t_0)$, the space-limiting function $s(\vec{x})$, and the external input field $q(\vec{x},t)$ are known. Thus, the first product in the integrand of equation (4-126) is known. In principle, the integration of this first term can be performed, yielding a known function of $\vec{x}$ and t. If we denote this known function by $h(\vec{x},t)$, that is,

$$h(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)s(\vec{x}_0)q(\vec{x}_0,t_0)\, d\vec{x}_0\, dt_0 \ , \qquad (4\text{-}141)$$

then equation (4-106) can be rewritten as

$$s(\vec{x})p(\vec{x},t) = h(\vec{x},t) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\vec{x},\vec{x}_0,t - t_0)\Sigma Q\{p(\vec{x}_0,t_0),\delta[\sigma(\vec{x}_0)]\}\, d\vec{x}_0\, dt_0 \ . \qquad (4\text{-}142)$$

Because $\delta[\sigma(\vec{x}_0)]$ is the derivative of the known function $s(\vec{x})$, the only unknown in equation (4-142) is the output field, $p(\vec{x},t)$, over all $\vec{x}$ and t. However, as $p(\vec{x},t)$ appears on both the left-hand side and in the integrand on the right-hand side, equation (4-142) represents an integral equation for the unrestricted output field, $p(\vec{x},t)$.

The treatment of such integral equations is beyond the scope of this text. Therefore, no attempt will be made to define or describe the wavevector-frequency characteristics of space-limited systems for which the output is specified by integral equations. The reader interested in such systems is encouraged to consult such standard texts as Morse and Feshbach[10] or Courant and Hilbert.[11] However, it should be emphasized that, in this text, we restrict our attention to space-limited systems for which the integrand of equation (4-126) is known. For such systems, the input-output relationships in the wavevector-frequency domain are given by equation (4-129) or (4-133), as appropriate.

Before we leave the subject of wavevector-frequency response of space-limited systems, a couple of observations are in order regarding the two-wavevector-frequency response, $G(\vec{k},\vec{\alpha},\omega)$, of space-limited systems.

First, recall that the physical interpretation of the two-wavevector-frequency response for space-limited systems is the same as that for the infinite, nonuniform type of space-varying system: that is, $G(\vec{k},\vec{\alpha},\omega)$ represents the response of the system at the wavevector $\vec{k}$ and frequency ω resulting from excitation of the system by a unit amplitude input at the

wavevector $-\vec{\alpha}$ and frequency ω. Recall further that, for the infinite, nonuniform system, it was (theoretically) possible, owing to the infinite extent of the system, to excite the system by an input characterized by a single wavevector and frequency component (i.e., a harmonic plane wave characterized by wavevector $-\vec{\alpha}$ and frequency ω). The resultant system output, in this case, defines the two-wavevector-frequency response of the system, $G(\vec{k},\vec{\alpha},\omega)$, over all wavevectors $\vec{k}$ for those (fixed) input parameters, $\vec{\alpha}$ and ω. It would be desirable to employ such a procedure to determine samplings of the two-wavevector-frequency response of the space-limited systems that one invariably encounters in practice. However, owing to the space-limited nature of the system, no such single wavevector-frequency excitation is possible. That is, for $F(\vec{k},\omega)$ to be a single wavevector-frequency component, $f(\vec{x},t)$ must be a harmonic plane wave, existing over all $\vec{x}$ and t. However, by equations (4-92) and (4-124), $f(\vec{x},t)$ is zero outside the spatial limits of the system, so excitation of a space-limited system by a single harmonic wave is impossible. Therefore, for space-limited systems, it is impractical to attempt direct measurement of the two-wavevector-frequency response of the system. Rather, common practice is to obtain spatial samples of the impulse response (or Green's function) as a function of time or frequency and, by discrete Fourier transformation of these spatial samples, obtain an estimate of the two-wavevector-frequency response.

The second observation regarding the two-wavevector-frequency response has to do with a distinction in terminology. Recall that the two-wavevector-frequency response defines the conversion, by the space-varying system, of each wavevector component of the input, at any frequency, to all wavevector components of the output at that same frequency. This wavevector conversion can result from either the space-varying properties of the system or from the boundaries of the system. In acoustics, it is common practice to refer to the wavevector conversion associated with system boundaries or abrupt discontinuities in system properties as wavevector scattering.

4.3.2.3 Summary of Wavevector-Frequency Response Characteristics of Space-Varying Systems. The space-varying systems treated in this text are limited to those in which all inputs (i.e., both external and boundary associated) to the system are known. We omit consideration of space-limited systems in which the output can only be specified as an integral equation.

For space-varying systems with known inputs, the relation between the wavevector-frequency descriptions of the input and output fields, $F(\vec{k},\omega)$ and $O(\vec{k},\omega)$, respectively, has the general mathematical form

$$O(\vec{k},\omega) = (2\pi)^{-3} \int_{-\infty}^{\infty} G(\vec{k},-\vec{\alpha},\omega)F(\vec{\alpha},\omega)\, d\vec{\alpha} \ . \tag{4-143}$$

Here, $G(\vec{k},\vec{\alpha},\omega)$ is the two-wavevector-frequency response of the system and defines the response of the system at the wavevector $\vec{k}$ and frequency ω resulting from a unit amplitude, plane wave input characterized by the wavevector $-\vec{\alpha}$ and frequency ω. Thus, the two-wavevector-frequency response is a metric of the conversion (or scattering), by the space-varying system, of each wavevector component $(-\vec{\alpha})$ of the input field, at any frequency, into the various wavevector components $(\vec{k})$ of the output field at that same frequency.

For nonuniform, infinite systems, $F(\vec{k},\omega)$ and $O(\vec{k},\omega)$, in equation (4-143), represent the wavevector-frequency transforms of the respective input and output space-time fields. These fields are infinite in spatial extent.

For space-limited systems, $O(\vec{k},\omega)$ represents the wavevector-frequency transform of the space-limited output field and, for systems with boundary conditions yielding an exact Green's function, $F(\vec{k},\omega)$ represents the wavevector-frequency transform of the space-limited input field. However, for systems having boundary conditions incompatible with the specification of an exact Green's function, $F(\vec{k},\omega)$ represents the wavevector-frequency transform of the sum of the space-limited input field and the additional space-time input field imposed by the constraints at the system boundaries.

4.3.3 Illustrative Examples of the Wavevector-Frequency Response of Space-Varying Systems

In this section, we present the wavenumber-frequency response of two space-limited systems having application to structural acoustics. These systems are (1) the acoustic field in an infinite half space resulting from excitation at the boundary and (2) the forced vibration of a simply supported, flat plate.

We first treat the problem of the acoustic half space.

4.3.3.1 <u>The Pressure Field in an Acoustic Half Space Excited at the Boundary</u>. A common problem in structural acoustics is the prediction of the acoustic field resulting from some specified displacement or velocity field at the boundary of the acoustic medium. Perhaps the most common version of this problem is the acoustic field produced in an infinite half space as a result of a known displacement field at the boundary of the half space.

Consider the acoustic half space depicted in figure 4-5. The half-space $x_3 > 0$ is occupied by a fluid of density ρ and speed of sound c. The space $x_3 < 0$ is vacuous. The displacement field on the plane $x_3 = 0$ is specified to be $w(\underline{x},t)$, where $\underline{x}$ denotes the two-dimensional vector (x_1,x_2). The consequent pressure field, $p(\vec{x},t)$, in the space $x_3 > 0$ is desired. The pressure in the space $x_3 < 0$ is, of course, zero.

The pressure in the half-space $x_3 > 0$ is governed by the homogeneous wave equation

$$\nabla^2 p(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2 p(\vec{x},t)}{\partial t^2} = 0 \ , \quad x_3 > 0 \ , \tag{4-144}$$

for all $\underline{x}$ and t. The linearized momentum equation for the acoustic fluid requires that, at the boundary $x_3 = 0$,

$$\frac{\partial p(\underline{x},0,t)}{\partial x_3} = -\rho \frac{\partial^2 w(\underline{x},t)}{\partial t^2} \ . \tag{4-145}$$

In addition, the pressure field must satisfy the causal condition that (because motion at the plane $x_3 = 0$ is responsible for the pressure field in the space $x_3 > 0$) the pressure must propagate away, or decay with increasing distance, from the boundary.

A Green's function solution for this space-limited acoustic field could be obtained by recognizing this system as a special case of the illustrative

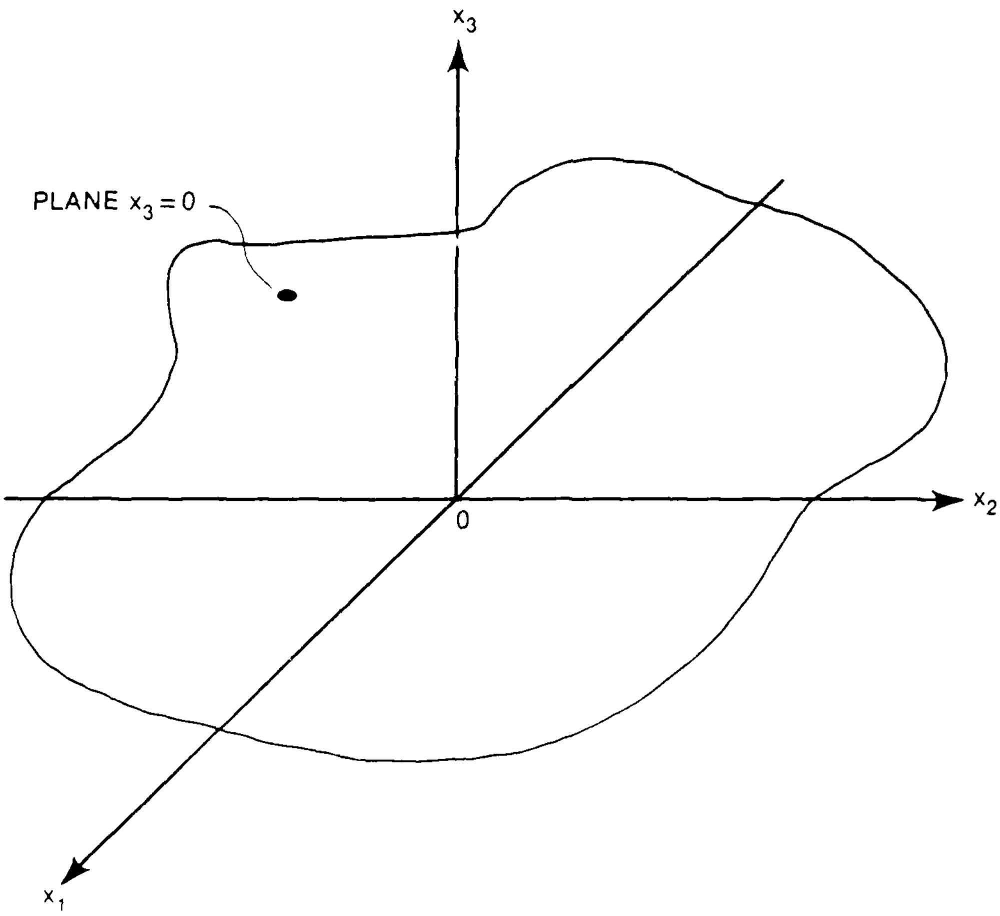

Figure 4-5. Geometry of the Acoustic Half Space

example presented in section 4.3.2.2, equations (4-134)-(4-139). In this special case, $q(\vec{x},t)$ in equation (4-134) is equal to zero inasmuch as there are no sources in the space $x_3 > 0$, and the boundary condition of equation (4-135) is replaced by that of equation (4-145). The Green's function is governed by equation (4-110) for $x_{30} \geq 0$. For $x_{30} < 0$, the Green's function is governed by a similar inhomogeneous wave equation, but with an initially unspecified distribution of sources on the right-hand side. This distribution of sources is then uniquely defined by requiring the Green's function to satisfy spatial and temporal causality and the restriction of equation (4-136).

A more direct and commonly used approach to this acoustic half-space problem is to solve the homogeneous wave equation (equation (4-144)) subject

to the boundary condition of equation (4-145). The wavevector-frequency description of the space-limited pressure field can then be obtained by appropriate Fourier transformation of the space-time field. This is the approach that will be presented here.

In light of the emphasis placed on Green's function solutions to space-varying systems in the previous sections, the reader is justified in asking why this direct, rather than the Green's function, approach is being adopted. The answer is that while the Green's function approach is most useful for developing the general mathematical forms of the input-output relationships for various types of systems and for introducing certain system concepts, it is not necessarily the simplest mathematical approach for obtaining a solution to a specific system. For the problem at hand, it is mathematically simpler to solve the homogeneous wave equation, subject to a boundary condition, than to solve the inhomogeneous wave equation, subject to a constraint on the input field. Whichever approach is taken, however, uniqueness demands that the solutions be mathematically equivalent.

With apologies for this lengthy preface, let us proceed with the solution for the problem of the acoustic half space, driven at the boundary.

The acoustic half space is invariant in time and in the two-dimensional spatial vector $\underline{x}$. Therefore, we assume that the pressure field can be written in the form

$$p(\underline{x},x_3,t) = (2\pi)^{-3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\underline{k},x_3,\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega \ , \quad x_3 > 0 \ , \tag{4-146}$$

where $\underline{k}$ denotes the two-dimensional wavevector (k_1,k_2). Substitution of equation (4-146) into (4-144) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{d^2P(\underline{k},x_3,\omega)}{dx_3^{\,2}} + (k_0^{\,2} - k^2)P(\underline{k},x_3,\omega)\right]\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega = 0 \tag{4-147}$$

for $x_3 > 0$ and for all $\underline{x}$ and t. In equation (4-147), k_0 denotes the acoustic wavenumber, ω/c, and k denotes $\sqrt{k_1^2 + k_2^2}$, the magnitude of the two-dimensional wavevector, $\underline{k}$. Inasmuch as equation (4-147) is valid for any choice of $\underline{x}$ or t, it follows that

$$\frac{d^2P(\underline{k},x_3,\omega)}{dx_3^2} + (k_0^2 - k^2)P(\underline{k},x_3,\omega) = 0 \ , \quad x_3 > 0 \ , \tag{4-148}$$

for all $\underline{k}$ and ω. The general solution of equation (4-148) is

$$P(\underline{k},x_3,\omega) = A(\underline{k},\omega)\exp\left\{i\sqrt{k_0^2 - k^2}\ x_3\right\} + B(\underline{k},\omega)\exp\left\{-i\sqrt{k_0^2 - k^2}\ x_3\right\} , \quad x_3 > 0 \ . \tag{4-149}$$

Substitution of equation (4-149) into equation (4-146) yields the following expression for the pressure field in the half-space $x_3 > 0$:

$$p(\underline{x},x_3,t) = (2\pi)^{-3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[A(\underline{k},\omega)\exp\left\{i[\underline{k}\cdot\underline{x} + \sqrt{k_0^2 - k^2}\ x_3 + \omega t]\right\}\right.$$

$$\left. + B(\underline{k},\omega)\exp\left\{i[\underline{k}\cdot\underline{x} - \sqrt{k_0^2 - k^2}\ x_3 + \omega t]\right\}\right] d\underline{k}\ d\omega \ . \tag{4-150}$$

The functions $A(\underline{k},\omega)$ and $B(\underline{k},\omega)$ in equations (4-149) and (4-150) are determined by application of (1) the causality, or radiation, condition that requires the pressure field to be comprised of waves which either propagate away, or decrease in amplitude with increasing distance, from the boundary $x_3 = 0$ and (2) the boundary condition of equation (4-145). Let us first examine the radiation condition.

At any fixed frequency, ω, wave components of the pressure field described by equation (4-150) that propagate away from $x_3 = 0$ in the positive x_3 direction are those characterized by the exponential forms

$$\exp\left\{i\left[\sqrt{k_0^2 - k^2}\ x_3 + \omega t\right]\right\} , \quad \omega < 0 ,$$

and

$$\exp\left\{i\left[-\sqrt{k_0^{\,2} - k^2}\; x_3 + \omega t\right]\right\} , \quad \omega \geq 0 ,$$

when $k \leq |k_0|$. It is also obvious, by inspection of equation (4-150), that waves which decay in amplitude with increasing distance in the positive x_3 direction from the boundary $x_3 = 0$ must be characterized by the exponential form

$$\exp\left\{-\sqrt{k^2 - k_0^{\,2}}\; x_3\right\} = \exp\left\{i\sqrt{k_0^{\,2} - k^2}\; x_3\right\} , \quad k > |k_0| .$$

For the pressure field described by equation (4-150) to be comprised only of waves consistent with these exponential forms, we require that

$$A(\underline{k},\omega) = 0 , \quad k \leq |k_0| \text{ and } \omega > 0 ,$$

$$B(\underline{k},\omega) = 0 , \quad k \leq |k_0| \text{ and } \omega < 0 , \tag{4-151}$$

$$B(\underline{k},\omega) = 0 , \quad k > |k_0| \text{ for all } \omega .$$

By defining

$$P_1(\underline{k},\omega) = \begin{Bmatrix} B(\underline{k},\omega) , & \omega \geq 0 \\ \\ A(\underline{k},\omega) , & \omega < 0 \end{Bmatrix} , \quad k \leq |k_0| , \tag{4-152}$$

and

$$P_2(\underline{k},\omega) = A(\underline{k},\omega) , \quad k > |k_0| \text{ for all } \omega , \tag{4-153}$$

we can express equation (4-150) in a form that satisfies the causality (or radiation) condition. That form is

$$p(\underline{x},x_3,t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \left[\int_{k \leq |k_0|} P_1(\underline{k},\omega) \exp\left\{ i\left[\underline{k}\cdot\underline{x} - k_0 x_3 \sqrt{1 - k^2/k_0^2} \right] \right\} d\underline{k} \right.$$

$$\left. + \int_{k > |k_0|} P_2(\underline{k},\omega) \exp\left\{ -\sqrt{k^2 - k_0^2}\, x_3 \right\} \exp\{ i\underline{k}\cdot\underline{x} \}\, d\underline{k} \right] \exp\{i\omega t\}\, d\omega \ . \tag{4-154}$$

We now employ the boundary condition of equation (4-145) to determine $P_1(\underline{k},\omega)$ and $P_2(\underline{k},\omega)$. This is most easily accomplished by first writing $w(\underline{x},t)$ in the form

$$w(\underline{x},t) = (2\pi)^{-3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\underline{k},\omega) \exp\{ i(\underline{k}\cdot\underline{x} + \omega t) \}\, d\underline{k}\, d\omega \ . \tag{4-155}$$

Then, by use of equations (4-154) and (4-155) in equation (4-145), we obtain

$$\int_{-\infty}^{\infty} \left\{ \int_{k \leq |k_0|} \left[\rho\omega^2 W(\underline{k},\omega) + ik_0 \sqrt{1 - k^2/k_0^2}\, P_1(\underline{k},\omega) \right] \exp\{ i(\underline{k}\cdot\underline{x} + \omega t) \}\, d\underline{k} \right.$$

$$\left. + \int_{k > |k_0|} \left[\rho\omega^2 W(\underline{k},\omega) + \sqrt{k^2 - k_0^2}\, P_2(\underline{k},\omega) \right] \exp\{ i(\underline{k}\cdot\underline{x} + \omega t) \}\, d\underline{k} \right\} d\omega = 0 \ , \tag{4-156}$$

which is valid for all $\underline{x}$ and t. It therefore follows that

$$P_1(\underline{k},\omega) = \frac{i\rho\omega^2 W(\underline{k},\omega)}{k_0 \sqrt{1 - k^2/k_0^2}}$$

and (4-157)

$$P_2(\underline{k},\omega) = \frac{-\rho\omega^2 W(\underline{k},\omega)}{\sqrt{k^2 - k_0^2}} \ .$$

Substitution of equations (4-157) into equation (4-154) yields the following expression for the space-time pressure field in the half-space $x_3 \geq 0$:

$$p(\underline{x}, x_3, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \left[\int_{k \leq |k_0|} \frac{i\rho\omega^2 W(\underline{k}, \omega)}{k_0 \sqrt{1 - k^2/k_0^2}} \exp\left\{ i[\underline{k} \cdot \underline{x} - k_0 x_3 \sqrt{1 - k^2/k_0^2} + \omega t] \right\} d\underline{k} \right.$$

$$\left. - \int_{k > |k_0|} \frac{\rho\omega^2 W(\underline{k}, \omega)}{\sqrt{k^2 - k_0^2}} \exp\left\{ -\sqrt{k^2 - k_0^2}\, x_3 \right\} \exp\{ i\underline{k} \cdot \underline{x} + \omega t \}\, d\underline{k} \right] d\omega \; . \tag{4-158}$$

The pressure in the half-space $x_3 < 0$, it will be recalled, is zero because that space is vacuous.

Comparison of equation (4-158) with equation (4-146) reveals that $P(\underline{k}, x_3, \omega)$ can be described in the half-space $x_3 \geq 0$ by

$$P(\underline{k}, x_3, \omega) = \begin{cases} \dfrac{i\rho\omega^2 W(\underline{k}, \omega)}{k_0 \sqrt{1 - k^2/k_0^2}} \exp\left\{ -ik_0 x_3 \sqrt{1 - k^2/k_0^2} \right\} , & k \leq |k_0| \\[2ex] \dfrac{-\rho\omega^2 W(\underline{k}, \omega)}{\sqrt{k^2 - k_0^2}} \exp\left\{ -\sqrt{k^2 - k_0^2}\, x_3 \right\} , & k > |k_0| \; . \end{cases} \tag{4-159}$$

Of course, $P(\underline{k}, x_3, \omega)$ is equal to 0 for $x_3 < 0$.

The complete wavevector-frequency transform of the pressure field in the half space is obtained by Fourier transformation of $\tilde{P}(\underline{k}, x_3, \omega)$ on the x_3 variable. This complete wavevector-frequency transform of the pressure field, denoted by $P(k, k_3, \omega)$, can be defined from equation (4-159) by use of the Heaviside function. That is,

$$\tilde{P}(\underline{k},k_3,\omega) = \begin{cases} \dfrac{i\rho\omega^2 W(\underline{k},\omega)}{k_0\sqrt{1 - k^2/k_0^2}} \displaystyle\int_{-\infty}^{\infty} U(x_3)\exp\left\{-i\left[k_3 + k_0\sqrt{1 - k^2/k_0^2}\right]x_3\right\} dx_3 \ , & k \le |k_0| \ , \\ \dfrac{-\rho\omega^2 W(\underline{k},\omega)}{\sqrt{k^2 - k_0^2}} \displaystyle\int_{-\infty} U(x_3)\exp\left\{-i\left[k_3 - i\sqrt{k^2 - k_0^2}\right]x_3\right\} dx_3 \ , & k > |k_0| \ . \end{cases} \tag{4-160}$$

The Fourier transform applicable to the wavenumber range $k \le |k_0|$ can be recognized as a Fourier transform of the Heaviside function $U(x_3)$. Papoulis[12] shows that

$$\int_{-\infty}^{\infty} U(x)\exp(-ikx)\, dx = \pi\delta(k) + 1/(ik) \ . \tag{4-161}$$

The transform applicable to the wavenumber range $k > |k_0|$ can be evaluated by simple integration so, by use of equation (4-161), it is straightforward to show that the complete wavevector-frequency transform of the pressure field in the half space is given by

$$\tilde{P}(\underline{k},k_3,\omega) = \begin{cases} -\dfrac{\rho\omega^2 W(\underline{k},\omega)}{k_0\sqrt{1 - k^2/k_0^2}} \left\{i\pi\delta\left[k_3 + k_0\sqrt{1 - k^2/k_0^2}\right] + \dfrac{1}{k_3 + k_0\sqrt{1 - k^2/k_0^2}}\right\} \ , & k \le |k_0| \ , \\ \dfrac{-\rho\omega^2 W(\underline{k},\omega)}{k^2 - k_0^2 + ik_3\sqrt{k^2 - k_0^2}} \ , & k > |k_0| \ . \end{cases} \tag{4-162}$$

Equations (4-159) and (4-162) represent two alternative forms by which a wavevector-frequency description of the pressure field can be related to the wavevector-frequency description of the displacement field at the boundary $x_3 = 0$. Equation (4-159) expresses the complex amplitudes of those waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ that comprise the pressure field on any surface of

constant x_3, in the range $0 \leq x_3 < \infty$, as a function of the complex amplitude of the corresponding wave component of the displacement field on the boundary. Equation (4-162) expresses the complex amplitudes of plane wave components of the form $\exp\{i(\underline{k} \cdot \underline{x} + k_3 x_3 + \omega t)\}$ that comprise the pressure field as a function of the complex amplitudes of the surface waves of the form $\exp\{i(\underline{k} \cdot \underline{x} + \omega t)\}$ that comprise the displacement field at the boundary. Before exploring the wavevector properties of these descriptions of the half-space pressure field, it is instructive to examine certain physical interpretations of these results.

Consider first the pressure field in the half-space $x_3 \geq 0$ resulting from an impulsive displacement, in space and time, at the boundary $x_3 = 0$. Let the pressure field resulting from this impulsive displacement field be denoted by $h(\underline{x},\underline{x}_0,x_3,t,t_0)$: that is,

$$p(\underline{x},x_3,t) = h(\underline{x},\underline{x}_0,x_3,t,t_0)$$

when

$$w(\underline{x},t) = \delta(\underline{x} - \underline{x}_0)\delta(t - t_0) \ . \tag{4-163}$$

By equations (4-155) and (4-158), it follows that, for $x_3 \geq 0$,

$$\begin{aligned} h(\underline{x},\underline{x}_0,x_3,t,t_0) = (2\pi)^{-3} \int_{-\infty}^{\infty} \Bigg[\int_{k \leq |k_0|} & \frac{i\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}} \\ & \exp\left\{i\left[\underline{k}\cdot(\underline{x} - \underline{x}_0) - k_0 x_3\sqrt{1 - k^2/k_0^2} + \omega(t - t_0)\right]\right\} d\underline{k} \\ & - \int_{k>|k_0|} \frac{\rho\omega^2}{\sqrt{k^2 - k_0^2}} \exp\left\{-\sqrt{k^2 - k_0^2}\, x_3\right\} \\ & \exp\{i[\underline{k}\cdot(\underline{x} - \underline{x}_0) + \omega(t - t_0)]\}\, d\underline{k} \Bigg] d\omega \ . \end{aligned} \tag{4-164}$$

For $x_3 < 0$, $h(\underline{x},\underline{x}_0,x_3,t,t_0)$ is equal to 0. By the form of the exponents in equation (4-164), it is evident that

$$h(\underline{x},\underline{x}_0,x_3,t,t_0) = h(\underline{x} - \underline{x}_0,x_3,t - t_0) \ . \tag{4-165}$$

This form of the impulse response is consistent with the invariance of the acoustic half-space system in the $\underline{x}$ and t variables and with the space-limited nature of the system in the x_3 coordinate.

By writing $W(\underline{k},\omega)$ in equation (4-158) as the multiple Fourier transform of $w(\underline{x}_0,t_0)$ and interchanging the order of integration, it is easily shown, by use of equation (4-164), that the pressure field in the half space is related to the displacement field at the boundary and the impulse response by

$$p(\underline{x},x_3,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\underline{x} - \underline{x}_0,x_3,t - t_0)w(\underline{x}_0,t_0)\, d\underline{x}_0\, dt_0 \ . \tag{4-166}$$

It is straightforward to show, from equation (4-166), that

$$P(\underline{k},x_3,\omega) = H(\underline{k},x_3,\omega)W(\underline{k},\omega) \tag{4-167}$$

and

$$\widetilde{P}(\underline{k},k_3,\omega) = \widetilde{H}(\underline{k},k_3,\omega)W(\underline{k},\omega) \ , \tag{4-168}$$

where H and $\widetilde{H}$ denote the two- and three-wavenumber-frequency transforms, respectively, of the impulse response, h. By comparison of equation (4-167) with equation (4-159), it is evident that, for $x_3 \geq 0$,

$$H(\underline{k},x_3,\omega) = \begin{cases} \dfrac{i\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}} \exp\left\{-ik_0\sqrt{1 - k^2/k_0^2}\; x_3\right\} , & k \leq |k_0| \ , \\[2ex] \dfrac{-\rho\omega^2}{\sqrt{k^2 - k_0^2}} \exp\left\{-\sqrt{k^2 - k_0^2}\; x_3\right\} , & k > |k_0| \ , \end{cases} \tag{4-169}$$

and $H(\underline{k},x_3,\omega) = 0$ for $x_3 < 0$. Similarly, comparison of equation (4-168) with equation (4-162) reveals that

$$\tilde{H}(\underline{k},k_3,\omega) = \begin{cases} \dfrac{\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}} \left\{ i\pi\delta\left[k_3 + k_0\sqrt{1 - k^2/k_0^2}\right] + \dfrac{1}{k_3 + k_0\sqrt{1 - k^2/k_0^2}} \right\}, & k \le |k_0| , \\[2ex] \dfrac{-\rho\omega^2}{k^2 - k_0^2 + ik_3\sqrt{k^2 - k_0^2}} , & k > |k_0| . \end{cases} \tag{4-170}$$

The quantity H can be interpreted, from equations (4-146) and (4-167), as the wavevector-frequency response of the acoustic half space to displacement at the boundary $x_3 = 0$. That is, by arguments similar to those used in section 3.4.3, $H(\underline{k},x_3,\omega)$ can be shown to represent the ratio of the space-time pressure field to the space-time displacement field at $x_3 = 0$ when that displacement field is a complex wave of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$. $\tilde{H}(\underline{k},k_3,\omega)$, the Fourier transform of $H(\underline{k},x_3,\omega)$ on the variable x_3, is simply the ratio of the complex amplitudes of the plane wave components of the form $\exp\{i(\underline{k}\cdot\underline{x} + k_3x_3 + \omega t)\}$ that comprise the pressure field to the complex amplitudes, at corresponding values of $\underline{k}$ and ω, of the waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ that comprise the displacement field at $x_3 = 0$.

One might be reasonably curious as to the relationship between the impulse response, $h(\underline{x} - \underline{x}_0,x_3,t - t_0)$, and the Green's function, $g(\vec{x},\vec{x}_0,t - t_0)$, for the semi-infinite acoustic system presented as an illustrative example in section 4.3.2.2 (see equation (4-137)). To specialize this illustrative example to the problem of the acoustic half space driven at the boundary, we first note that, inasmuch as the half space is driven only at the boundary, no sources are present in the space $x_3 > 0$, and thereby $q(\vec{x},t) = 0$. By equation (4-134), this implies that $f(\vec{x},t) = 0$. Further, from the boundary condition for the acoustic half space (equation (4-145)), it follows that $a(\underline{x},t)$ in equation (4-135) is given by

$$a(\underline{x},t) = -\rho \frac{\partial^2 w(\underline{x},t)}{\partial t^2} . \tag{4-171}$$

With these conditions applied to equation (4-137), the Green's function solution for the acoustic half space driven at the boundary can be written

$$o(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\underline{x},x_3;\underline{x}_0,0;t - t_0)\ \rho\ \frac{\partial^2 w(\underline{x}_0,t_0)}{\partial t_0{}^2}\ d\underline{x}_0\ dt_0 . \tag{4-172}$$

By recognizing that the acoustic half-space system is invariant in the two-dimensional spatial vector, $\underline{x}$, and by using equation (4-134), it follows that equation (4-172) has the form

$$U(x_3)p(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\underline{x} - \underline{x}_0;x_3,0;t - t_0)\ \rho\ \frac{\partial^2 w(\underline{x}_0,t_0)}{\partial t_0{}^2}\ d\underline{x}_0\ dt_0 . \tag{4-173}$$

As the only x_3 variation in the integrand of equation (4-173) is that associated with the Green's function, it follows that the Green's function must be of the mathematical form

$$g(\underline{x} - \underline{x}_0;x_3,0;t - t_0) = U(x_3)g'(\underline{x} - \underline{x}_0;x_3,0;t - t_0) , \tag{4-174}$$

where g' is a function equal to g in the half-space $x_3 \geq 0$, but of arbitrary specification in the space $x_3 < 0$.

The product of the fluid density (ρ) and the second derivative of the boundary displacement (w) with respect to time in the integrand of equation (4-173) can be interpreted as the inertial force of each unit volume of fluid at the boundary. It follows that the Green's function, $g(\underline{x} - \underline{x}_0;x_3,0;t - t_0)$, can be interpreted as the pressure field resulting from an impulsive inertial force applied to a unit volume of fluid at the boundary $x_3 = 0$. This interpretation is in contrast to that of the impulse response $h(\underline{x} - \underline{x}_0;x_3;t - t_0)$, defined by equation (4-163), which represents the pressure field resulting from an impulsive displacement at the boundary.

It is straightforward to show, from equations (4-155) and (4-173), that

$$U(x_3)P(\underline{k},x_3,\omega) = -G(\underline{k};x_3,0;\omega)\{\rho\omega^2 W(\underline{k},\omega)\} \quad . \tag{4-175}$$

Further, by comparison of equations (4-167) and (4-175), it is evident that

$$G(\underline{k};x_3,0;\omega) = -H(\underline{k},x_3,\omega)/[\rho\omega^2] \quad . \tag{4-176}$$

Clearly then, the wavevector-frequency response associated with the Green's function is related to the wavevector-frequency response associated with the impulse response, $h(\underline{x} - \underline{x}_0;x_3;t - t_0)$. By substitution of equation (4-169) in equation (4-176), an expression for the Green's function, $g(\underline{x} - \underline{x}_0;x_3,0;t - t_0)$, can be obtained in terms of an inverse Fourier transform. This procedure is left as an exercise for the interested reader.

If we denote the three-wavenumber-frequency transform of the Green's function by $\tilde{G}(\underline{k};k_3,0;\omega)$, it follows from equation (4-176) that

$$\tilde{G}(\underline{k};k_3,0;\omega) = -\tilde{H}(\underline{k},k_3,\omega)/[\rho\omega^2] \quad . \tag{4-177}$$

As a final note on the Green's function, it follows from equations (4-134) and (4-175) that

$$O(\underline{k},k_3,\omega) = -\tilde{G}(\underline{k};k_3,0;\omega)\{\rho\omega^2 W(\underline{k},\omega)\} \quad . \tag{4-178}$$

By comparison of equation (4-178) with equation (4-139) (with $F(\vec{\alpha},\omega)$ set to zero as no sources are present in the space $x_3 > 0$ and with $A(\underline{\alpha},\omega) = \rho\omega^2 W(\underline{\alpha},\omega)$), there appears to be a difference in the two solutions. However, owing to the invariance of the acoustic half-space system in the two-dimensional spatial vector $\underline{x}$ and in time, it may be shown that $G(\vec{k},-\vec{\alpha},\omega)$ in equation (4-139) takes the form

$$G(\vec{k},-\vec{\alpha},\omega) = (2\pi)^2 \delta(\underline{\alpha} - \underline{k})\overset{\nabla}{G}(\underline{k},k_3;\alpha_3,\omega) \quad , \tag{4-179}$$

where $\overset{\nabla}{G}$ is the multiple Fourier transform of $g(\underline{x} - \underline{x}_0;x_3,x_{30};t - t_0)$ on the variables $\underline{x} - \underline{x}_0$, x_3, x_{30}, and $t - t_0$. By substituting equation (4-179) into equation (4-139) and performing the integration on $\underline{\alpha}$, we obtain

$$O(\underline{k},k_3,\omega) = -\rho\omega^2 W(\underline{k},\omega)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overset{\nabla}{G}(\underline{k},k_3;\alpha_3;\omega)\,d\alpha_3\right\} . \qquad (4\text{-}180)$$

However, the integral in equation (4-180) can easily be shown to be equal to $\tilde{G}(\underline{k};k_3,0;\omega)$. Thus, the result of equation (4-178) is not in conflict with, but rather a consequence of, equation (4-139).

In some of the literature dealing with structural acoustics (see, for example, reference 13), the pressure field in the acoustic half space is expressed in terms of the spectral surface impedance of the acoustic medium. The spectral surface impedance, Z_s, is defined as the ratio of the wavevector-frequency transform of the pressure field at the boundary $x_3 = 0$ to the wavevector-frequency transform of the normal velocity of the boundary. The wavevector-frequency transform of the pressure field at the boundary is specified by equation (4-159), evaluated at $x_3 = 0$. The normal velocity at the boundary, $v(\underline{x},t)$, is the temporal derivative of the displacement field, $w(x,t)$. By use of equation (4-155), it is easily shown that the wavevector-frequency transform of the velocity field, $V(\underline{k},\omega)$, is related to the wavevector-frequency transform of the displacement field by $V(\underline{k},\omega) = i\omega W(\underline{k},\omega)$. It therefore follows that the spectral surface impedance is given by

$$Z_s(\underline{k},\omega) = P(\underline{k},0,\omega)/V(\underline{k},\omega) = P(\underline{k},0,\omega)/[i\omega W(\underline{k},\omega)] . \qquad (4\text{-}181)$$

A concept often used in conjunction with the spectral surface impedance in acoustics is that of the spectral transfer function. The spectral transfer function of the pressure field in the half space, denoted by $T(\underline{k},x_3,\omega)$, is the ratio of the wavevector-frequency transform of the pressure field at a distance x_3 from the boundary to the wavevector-frequency transform of the pressure field at the boundary. That is,

$$T(\underline{k},x_3,\omega) = \frac{P(\underline{k},x_3,\omega)}{P(\underline{k},0,\omega)} . \qquad (4\text{-}182)$$

It follows, by equations (4-181) and (4-182), that the wavevector-frequency transform of the pressure field in the half space is related to the spectral surface impedance and the spectral transfer function by

$$P(\underline{k},x_3,\omega) = Z_s(\underline{k},\omega)T(\underline{k},x_3,\omega)V(\underline{k},\omega) \quad . \tag{4-183}$$

By comparison of equation (4-167) with equation (4-181), the spectral surface impedance is related to the wavevector-frequency response, $H(\underline{k},x_3,\omega)$, by

$$Z_s(\underline{k},\omega) = H(\underline{k},0,\omega)/(i\omega) \quad . \tag{4-184}$$

Further, by equations (4-167) and (4-182), the spectral transfer function is related to the wavevector-frequency response by

$$T(\underline{k},x_3,\omega) = \frac{H(\underline{k},x_3,\omega)}{H(\underline{k},0,\omega)} \quad . \tag{4-185}$$

By the above arguments, it is evident that the wavevector-frequency transform of the pressure field in the half space can be expressed in terms of several different, but related, descriptors. These various descriptors relate the wavevector-frequency description of the pressure field to wavevector-frequency descriptions of different physical characterizations of the excitation applied at the boundary. The selection of any particular descriptor for the wavevector-frequency analysis of acoustic fields is usually made for reasons of mathematical convenience or personal preference.

Figure 4-6 illustrates the magnitude and phase of the wavevector-frequency response, $H(\underline{k},x_3,\omega)$, of the acoustic half space for an arbitrary, but positive, frequency at three values of the dimensionless spatial variable k_0x_3. Recall, by equation (4-166), that $H(\underline{k},x_3,\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ is the pressure field that results from the displacement field $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ applied at the boundary, $x_3 = 0$.

To aid in the interpretation of figure 4-6, it should first be noted, from equation (4-169), that the argument (or phase) of $H(\underline{k},x_3,\omega)$ is given by

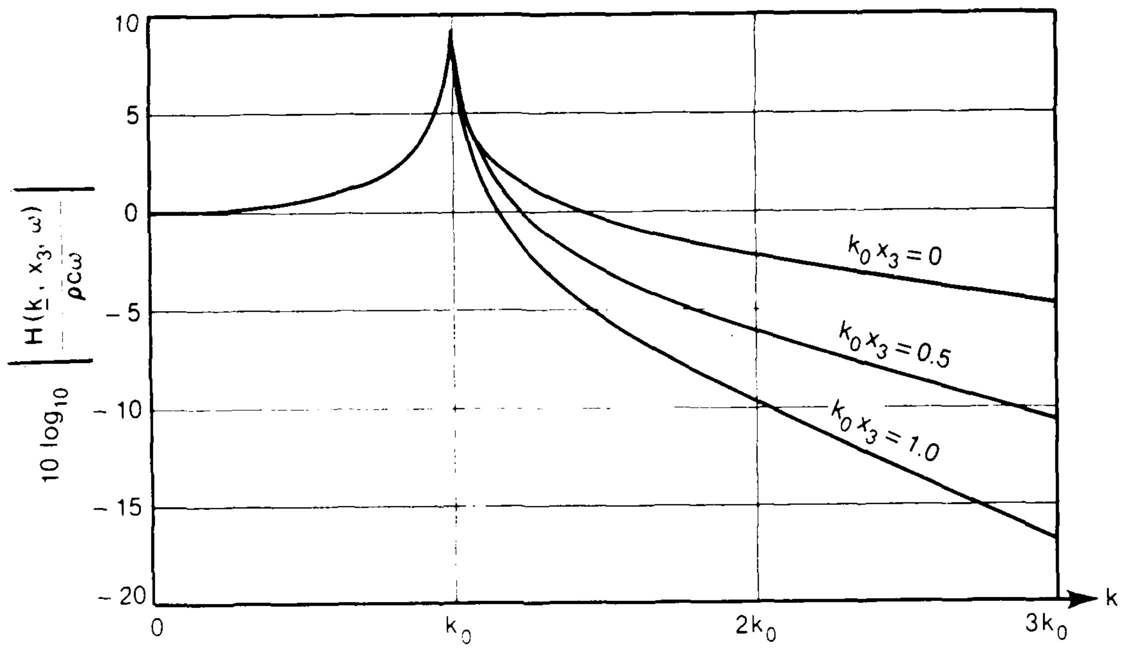

Figure 4-6(a). Magnitude of $H(\underline{k}, x_3, \omega)$ as a Function of k

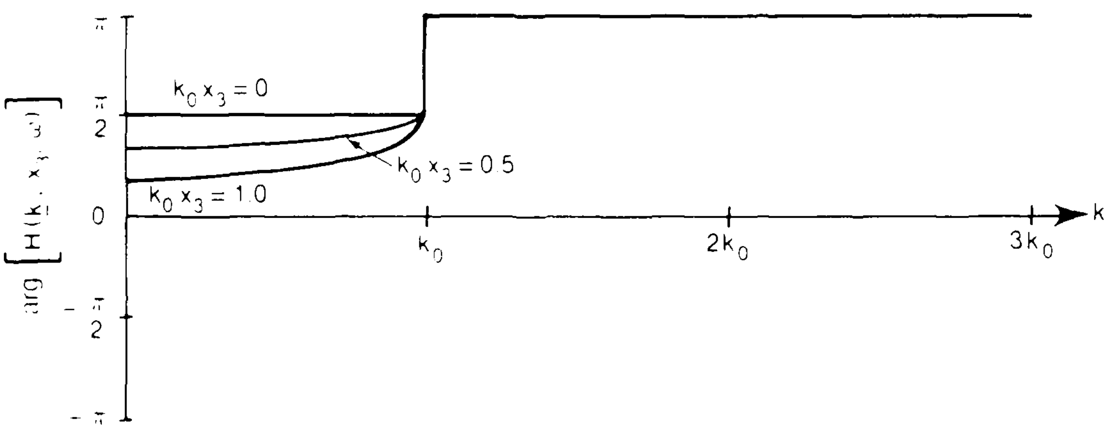

Figure 4-6(b). Phase of $H(\underline{k}, x_3, \omega)$ as a Function of k

Figure 4-6. Magnitude and Phase of $H(k, x_3, \omega)$ as a Function of k

$$\arg\{H(\underline{k},x_3,\omega)\} = \begin{cases} \operatorname{sgn}(\omega)\,\frac{\pi}{2} - k_0\sqrt{1 - k^2/k_0^2}\;x_3\,, & k \le |k_0|\,, \\ \pi\,, & k > |k_0|\,. \end{cases} \tag{4-186}$$

Therefore, it follows that the pressure field resulting from the displacement field $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ applied at the boundary can be written

$$p(\underline{x},x_3,t) = \begin{cases} |H(\underline{k},x_3,\omega)|\;\exp\left\{i\left[\underline{k}\cdot\underline{x} - k_0x_3\sqrt{1 - k^2/k_0^2} + \omega t + (\pi/2)\operatorname{sgn}(\omega)\right]\right\}\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \le |k_0|\,, \\ |H(\underline{k},x_3,\omega)|\;\exp\{i(\underline{k}\cdot\underline{x} + \omega t) + \pi\}\,, \qquad k > |k_0|\,. \end{cases} \tag{4-187}$$

By equation (4-187), it is evident that, if the magnitude of the wavevector, $\underline{k}$, characterizing the displacement field, $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$, is less than or equal to the magnitude of the acoustic wavenumber, $k_0 = \omega/c$, of the fluid medium, the pressure field in the half space is a plane wave of amplitude $|H(\underline{k},x_3,\omega)|$, which is characterized by the wavevector $\underline{k}$ in the plane of the boundary and by the wavevector $-k_0\sqrt{1 - k^2/k_0^2}$ in the x_3 coordinate direction. Therefore, on a plane parallel to the boundary, this wave propagates in the same direction and with the same phase speed as the displacement wave at the boundary. In the x_3 coordinate direction, the wave propagates away from the boundary at an angle Θ to the plane of the boundary. The angle Θ is given by

$$\Theta = \arctan\left\{\frac{|k_0|\sqrt{1 - k^2/k_0^2}}{k}\right\}. \tag{4-188}$$

The phase speed, c_3, of the wave in the positive x_3 direction is given by

$$c_3 = \frac{c}{\sqrt{1 - k^2/k_0^2}}\;. \tag{4-189}$$

When the magnitude of the wavevector characterizing the displacement of the boundary is greater than the magnitude of k_0, equation (4-187) reveals that the pressure field in the half space, at any positive value of x_3, is characterized by a wave of the same form as that applied at the boundary: that

is, by $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$. However, the pressure wave is 180 degrees out of phase with the displacement wave. The amplitude of the pressure wave is specified by $|H(\underline{k},x_3,\omega)|$ and, according to equation (4-169), decreases exponentially with increasing distance (x_3) from the boundary. As the pressure wave is characterized, on any plane of constant and positive x_3, by the same wavevector and frequency as the displacement wave at the boundary, the wave propagates on that plane in the same direction and with the same phase speed as does the wave on the boundary. These waves that propagate parallel to the boundary, but decrease exponentially in amplitude with distance from the boundary, are called evanescent waves.

Figure 4-6(a) presents the magnitude of $H(\underline{k},x_3,\omega)$ normalized by $\rho c\omega$, the value of $H(\underline{k},x_3,\omega)$ at $\underline{k} = (0,0)$, as a function k (the magnitude of $\underline{k}$). From the above discussion, the magnitude of $H(\underline{k},x_3,\omega)$ represents (1) the amplitudes of the acoustic waves that propagate in x_3 in the wavenumber range $k \leq |k_0|$ and (2) the amplitudes of the evanescent waves that only propagate along surfaces of constant (and positive) x_3 in the wavenumber range $k > |k_0|$. The normalized magnitude of $H(\underline{k},x_3,\omega)$ is presented for three values of the normalized coordinate k_0x_3: 0, 0.5, and 1.

Note first that, because the acoustic half-space system is space invariant in the two-dimensional vector variable $\underline{x}$, the wavevector-frequency response, $H(k,x_3,\omega)$, is a function of only the magnitude of the two-dimensional wavevector $\underline{k}$. That is, owing to the spatial invariance in $\underline{x}$, the response of the half space to a wave of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ at the boundary is independent of the direction of propagation of the applied wave.

Note further that, in the wavenumber range $k \leq |k_0|$, the magnitude of $H(\underline{k},x_3,\omega)$, and thereby the amplitude of the acoustic wave radiated into the half space, is independent of the variable x_3. However, as is evident by figure 4-6(a), the amplitude of the evanescent waves associated with the wavenumber range $k > |k_0|$ decreases with increasing positive values of x_3 and with increasing values of k.

Note that when the magnitude of the wavevector that characterizes the wave of displacement, $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$, at the boundary is equal to $|k_0|$, the

magnitude of $H(\underline{k},x_3,\omega)$, and thereby the amplitude of the radiated pressure field in the half space, becomes infinite. However, if the pressure field had infinite amplitude at the boundary, a force of infinite amplitude would be required to initiate any displacement of the boundary. Thus, the infinite value of $|H(\underline{k},x_3,\omega)|$ at $k = |k_0|$ is best interpreted as a statement that the surface impedance of the acoustic half space becomes infinite at $k = |k_0|$, and thus no wave of displacement (or, more properly, velocity) characterized by such a wavevector magnitude can be excited at the surface.

Figure 4-6(b) illustrates the argument, or phase, of $H(\underline{k},x_3,\omega)$ as a function of the magnitude of the wavevector characterizing the wave of displacement, $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$, on the bounding surface, $x_3 = 0$. In the wavenumber range $k \leq |k_0|$, where acoustic propagation occurs in the x_3 coordinate direction, equation (4-186) shows the phase to depend on both the magnitude of the wavevector of excitation, k, and the distance from the boundary, x_3. Figure 4-6(b) illustrates the wavenumber dependency for k_0x_3 equal to 0, 0.5, and 1. For wavevectors of excitation greater, in magnitude, than $|k_0|$, the phase of $H(\underline{k},x_3,\omega)$ is independent of k and equal to π.

The wavevector characteristics of $\tilde{H}(\underline{k},k_3,\omega)$, the complete wavevector-frequency transform of $h(\underline{x} - \underline{x}_0,x_3,t - t_0)$ defined by equation (4-170), are difficult to illustrate in graphical form. However, they can be described and interpreted.

Recall that $\tilde{H}(\underline{k},k_3,\omega)$ is the ratio of the complex amplitudes of waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + k_3x_3 + \omega t)\}$ that comprise the pressure field in the space $x_3 \geq 0$ to the complex amplitudes, at corresponding wavevectors ($\underline{k}$) and frequencies (ω), of waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ that comprise the displacement field on the boundary. We have established that a wavevector component, $\underline{k}$, of the displacement field produces a plane wave of pressure in the half-space $x_3 > 0$ when $|\underline{k}| \leq |k_0|$. The propagation of that plane wave in x is characterized, for all x_3, by the wavevector, $\underline{k}$, and frequency, ω, of the displacement field. The propagation in the x_3 coordinate direction, however, is determined by the allowable waves (i.e., the free waves) in the acoustic medium and the radiation condition that waves in the acoustic half

space must propagate in the positive x_3 direction (i.e., away from the boundary).

It is straightforward to establish, by arguments similar to those used in section 3.3, that free waves in an infinite acoustic medium are governed, in the wavevector-frequency domain, by

$$[k_1^2 + k_2^2 + k_3^2 - (\omega/c)^2]P(\underline{k},k_3,\omega) = [k^2 + k_3^2 - k_0^2]P(\underline{k},k_3,\omega) = 0 \ . \tag{4-190}$$

For fixed values of k_1, k_2, and ω, it follows that the wavevector-frequency description of allowable free waves in an infinite acoustic medium is of the form

$$P(\underline{k},k_3,\omega) = A(\underline{k},\omega)\delta\left\{k_3 - k_0\sqrt{1 - k^2/k_0^2}\right\} + B(\underline{k},\omega)\delta\left\{k_3 + k_0\sqrt{1 - k^2/k_0^2}\right\} . \tag{4-191}$$

The first term on the right-hand side corresponds to a wave propagating in the *negative* x_3 direction, and the second term corresponds to a wave propagating in the positive x_3 direction.

Note that, for $k \leq |k_0|$, the first term on the right-hand side of equation (4-170) is of the form of the second term on the right-hand side of equation (4-191) and therefore corresponds to a free acoustic wave propagating in the positive x_3 direction in an infinite acoustic medium. The values of k and ω associated with this wave are dictated by the wavevector-frequency component of interest in the displacement field. The amplitude of this free wave is seen, in equation (4-170), to be a function of the wavenumber and frequency of the boundary excitation.

The acoustic half space, however, is not an infinite acoustic medium; it is space limited in the x_3 coordinate. Therefore, in the wavenumber range $k < |k_0|$, the free wave must be augmented by other wave components to eliminate the pressure field in the space $x_3 < 0$. The second term on the right-hand side of that portion of equation (4-170) applicable to the wavenumber range $k \leq |k_0|$ defines these additional wave components. For a

given wavevector-frequency component of the displacement field (and thereby given values of k and ω), it is evident that the largest of these additional components occurs at $k_3 = -\sqrt{k_0^2 - k^2}$.

For a wave component of the displacement field characterized by a wavevector and frequency such that $k > |k_0|$, we have established that the resulting pressure field, on any plane of constant and positive x_3, propagates in the same direction and with the same speed as the displacement field. However, the amplitude of this pressure wave decreases exponentially with increasing x_3. In the wavenumber range $k > |k_0|$, equation (4-170) defines the complex amplitudes of that combination of plane waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + k_3x_3 + \omega t)\}$ that produces such an evanescent wave field for each corresponding wavevector, $\underline{k}$, and frequency, ω, component of the displacement field on the boundary. In this wavenumber range, it is easily established from equation (4-170) that the magnitude of $\tilde{H}(\underline{k},k_3,\omega)$ is inversely proportional to $k^2 + k_3^2 - k_0^2$ and is therefore largest when the magnitude of the wavevector $\vec{k} = (k_1,k_2,k_3)$ is greater than, but in the neighborhood of, $|k_0|$: that is, when

$$\sqrt{k^2 + k_3^2} = \sqrt{k_1^2 + k_2^2 + k_3^2} \approx |k_0| .$$

While it is not obvious how the distribution of plane waves in equation (4-170) produces a pressure field that decays in amplitude with increasing positive values of x_3 when $k > |k_0|$, it should be noted that, in the complex k_3 plane, $\tilde{H}(\underline{k},k_3,\omega)$ is characterized by a simple pole on the positive, imaginary k_3 axis. By the Cauchy integral theorem,[14] this pole corresponds, under Fourier transformation of $\tilde{H}$ on k_3, to the exponential decay noted in $H(\underline{k},x_3,\omega)$ for positive x_3 and to zero for negative x_3.

To summarize, we have presented two alternative wavevector-frequency descriptions of the pressure field produced in an acoustic half space by a prescribed displacement field at the boundary. This acoustic system is space invariant in the two-dimensional space $\underline{x} = (x_1,x_2)$, but is space limited in x_3.

One wavevector-frequency description is the Fourier transform of the pressure field on only those variables over which the system is invariant:

that is, the spatial vector variable $\underline{x}$ and time. In this case, the wavevector-frequency transform of the pressure field, $P(\underline{k},x_3,\omega)$, was shown to be equal to the product of the corresponding transform of the displacement field at the boundary, $W(\underline{k},\omega)$, and the wavevector-frequency response of the acoustic half space, $H(\underline{k},x_3,\omega)$. This wavevector-frequency response was shown to be related to the spectral surface impedance, $Z(\underline{k},\omega)$, and to the corresponding wavevector-frequency transform of the Green's function, $G(\underline{k};x_3,0,\omega)$. The magnitude of the wavevector-frequency response (and that of these related descriptors of the system response) was shown to be greatest, for all $x_3 \geq 0$, at those wavevectors equal, in magnitude, to the free wavevector, k_0, of the acoustic medium.

The second wavevector-frequency description of the pressure field, $\tilde{P}(\underline{k},k_3,\omega)$, was formed by Fourier transformation of the space-time pressure field on all independent variables (i.e., on $\underline{x}$, x_3, and t). This transform of the pressure field was shown to be equal to the product of the wavevector-frequency transform of the boundary displacement field, $W(\underline{k},\omega)$, and the Fourier transform of the wavevector-frequency response, $H(\underline{k},x_3,\omega)$, of the acoustic half space on the spatial variable x_3. This complete wavevector-frequency transform of the displacement impulse response, $h(\underline{x} - \underline{x}_0,x_3,t - t_0)$, was denoted by $\tilde{H}(\underline{k},k_3,\omega)$ and was shown to represent the ratio of the complex amplitudes of the plane wave components of the form $\exp\{i(\underline{k}\cdot\underline{x} + k_3x_3 + \omega t)\}$ comprising the pressure field to the complex amplitudes, at corresponding values of $\underline{k}$ and ω, of the waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ comprising the displacement field at the boundary. It was shown that this complete wavevector-frequency response, $\tilde{H}(\underline{k},k_3,\omega)$, was characterized by discrete wavevector contributions (i.e., delta functions) on the hemisphere defined by $k^2 + k_3^2 = k_1^2 + k_2^2 + k_3^2 = k_0^2$, $k_3 \leq 0$. Further, the magnitude of $\tilde{H}(\underline{k},k_3,\omega)$ was shown to approach infinity when the magnitude of the wavevector $\vec{k} = (k_1,k_2,k_3)$ approached the magnitude of the free wavenumber of the acoustic medium, k_0.

The point to be stressed is that the magnitude of the wavevector-frequency response, in either the two- or three-dimensional wavevector form, of this boundary excited, space-varying acoustic system is greatest at wavevectors equal, in magnitude, to that of the free wavenumber of the acoustic medium.

Recall, from chapter 3, that the magnitude of the wavevector-frequency response of space- and time-invariant systems was also found to be greatest at wavevectors equal, in magnitude, to that of the free wavenumber of the system.

4.3.3.2 The Forced Vibration of a Simply Supported Plate. The free vibration of a simply supported plate was treated in section 4.2.2 to illustrate the wavevector-frequency properties of a free system, space limited in two dimensions. Here, to illustrate the wavevector-frequency properties of a forced system that is space limited in two dimensions, we investigate the forced vibration of the simply supported plate.

In this example, the simply supported plate illustrated in figure 4-3 is subjected to a force per unit area, $f(\underline{x},t)$, that is considered positive when it acts in the direction of positive displacement, $w(\underline{x},t)$, of the plate. To simplify temporal causality arguments, the plate is subjected to a damping force per unit area equal to $r\partial w/\partial t$, which opposes motion of the plate. By using the notation of section 4.2.2, the displacement field of the plate resulting from the externally applied forcing field is governed, over $0 < x_1 < L_1$, $0 < x_2 < L_2$, and all t, by

$$\beta(\underline{x})\left\{D^4 w(\underline{x},t) + r\,\frac{\partial w(\underline{x},t)}{\partial t} + \mu\,\frac{\partial^2 w(\underline{x},t)}{\partial t^2}\right\} = \beta(\underline{x})f(\underline{x},t)\ , \tag{4-192}$$

where $\beta(\underline{x})$ is the two-dimensional space-limiting function defined by equation (4-46). The displacement field in the space outside the physical extent of the plate is assumed to be zero. At the boundaries of the plate, the displacement field must satisfy the simply supported conditions specified by equations (4-48) and (4-49).

The simply supported plate is a causal, time-invariant system, and it was established in section 4.2.2 that the normal modes of the plate, $\alpha_{mn}(\underline{x})$, defined by equation (4-53) individually satisfy the simply supported boundary conditions. We therefore assume that the displacement field can be expressed in the form

$$w(\underline{x},t) = (2\pi)^{-1}\int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(\omega)\alpha_{mn}(\underline{x})\exp(i\omega t)\ d\omega \tag{4-193}$$

over $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$ for all t. We also assume the forcing field, $f(\underline{x},t)$, can be expressed by

$$f(\underline{x},t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} B_{qs}(\omega)\alpha_{qs}(\underline{x})\exp(i\omega t)\, d\omega \qquad (4\text{-}194)$$

over $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$ for all t. Substitution of equations (4-193) and (4-194) into equation (4-192) yields

$$\int_{-\infty}^{\infty} \Bigg[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2\} A_{mn}(\omega)\beta(\underline{x})\alpha_{mn}(\underline{x})$$

$$- \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} B_{qs}(\omega)\beta(\underline{x})\alpha_{qs}(\underline{x}) \Bigg] \exp(i\omega t)\, d\omega = 0 \ . \qquad (4\text{-}195)$$

As equation (4-195) is valid for all t, it follows that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2\} A_{mn}(\omega)\beta(\underline{x})\alpha_{mn}(\underline{x})$$

$$- \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} B_{qs}(\omega)\beta(\underline{x})\alpha_{qs}(\underline{x}) = 0 \ . \qquad (4\text{-}196)$$

By multiplying equation (4-196) by $\alpha_{uv}(\underline{x})$ and integrating over all $\underline{x}$, we can use the orthogonality condition of equation (4-54) to show that

$$A_{mn}(\omega) = \frac{B_{mn}(\omega)}{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2} \ . \qquad (4\text{-}197)$$

However, from equation (4-194) and the orthogonality condition of equation (4-54), it can be shown that

$$B_{mn}(\omega) = (4/L_1L_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\underline{x},t)\beta(\underline{x})\alpha_{mn}(\underline{x})\exp(-i\omega t)\, d\underline{x}\, dt \ . \qquad (4\text{-}198)$$

Therefore, by equations (4-193), (4-197), and (4-198), it follows that

$$w(\underline{x},t) = \frac{2}{\pi L_1 L_2} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\underline{z},\Theta)\beta(\underline{z})\alpha_{mn}(\underline{z})\exp(-i\omega\Theta)d\underline{z}d\Theta}{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2} \right\}$$

$$\alpha_{mn}(\underline{x})\exp(i\omega t)\, d\omega \tag{4-199}$$

over $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$ for all t. Recall that $w(\underline{x},t) = 0$ for $\underline{x}$ outside $0 \le x_1 \le L_1$ or $0 \le x_2 \le L_2$.

To obtain a single expression for the displacement field, valid over all space and time, it is convenient to define the field $w_\infty(\underline{x},t)$ as the extension of equation (4-199) over all space. That is,

$$w_\infty(\underline{x},t) = \frac{2}{\pi L_1 L_2} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\underline{z},\Theta)\beta(\underline{z})\alpha_{mn}(\underline{z})\exp(-i\omega\Theta)d\underline{z}d\Theta}{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2} \right\}$$

$$\alpha_{mn}(\underline{x})\exp(i\omega t)\, d\omega \tag{4-200}$$

for all $\underline{x}$ and t. The displacement field that, for all time, satisfies equation (4-192) in the space $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$, the boundary conditions of equations (4-48) and (4-49), and the requirement of zero displacement for $\underline{x}$ outside $0 \le x_1 \le L_1$ or $0 \le x_2 \le L_2$ can then be written

$$w(\underline{x},t) = \beta(\underline{x})w_\infty(\underline{x},t) \; . \tag{4-201}$$

By definition, the Green's function for the simply supported plate is a solution to equation (4-192) when $f(\underline{x},t)$ is an impulse in time and space. That is, the Green's function is governed by

$$\beta(\underline{x})\left\{D\nabla^4 g(\underline{x},\underline{x}_0,t,t_0) + r\,\frac{\partial g(\underline{x},\underline{x}_0,t,t_0)}{\partial t} + \mu\,\frac{\partial^2 g(\underline{x},\underline{x}_0,t,t_0)}{\partial t^2}\right\}$$

$$= \beta(\underline{x})\delta(\underline{x},\underline{x}_0)\delta(t - t_0) \; . \tag{4-202}$$

To complete the specification of the Green's function, suitable spatial and temporal constraints must be applied to the solutions to equation (4-202).

If we require that the Green's function satisfy the simply supported conditions at the boundaries of the plate, that is,

$$g(0,x_2;\underline{x}_0;t,t_0) = g(L_1,x_2;\underline{x}_0;t,t_0) = g(x_1,0;\underline{x}_0;t,t_0) = g(x_1,L_2;\underline{x}_0;t,t_0) = 0 \tag{4-203}$$

and

$$\frac{\partial^2 g(0,x_2;\underline{x}_0;t,t_0)}{\partial x_1^{\,2}} = \frac{\partial^2 g(L_1,x_2;\underline{x}_0;t,t_0)}{\partial x_1^{\,2}}$$

$$= \frac{\partial^2 g(x_1,0;\underline{x}_0;t,t_0)}{\partial x_2^{\,2}} = \frac{\partial^2 g(x_1,L_2;\underline{x}_0;t,t_0)}{\partial x_2^{\,2}} = 0 \ , \tag{4-204}$$

then the form of the Green's function can be obtained from equations (4-200) and (4-201) by replacing $f(\underline{x},t)$ by $\delta(\underline{x},\underline{x}_0)\delta(t - t_0)$. That form is easily shown to be

$$g(\underline{x},x_0,t - t_0) = \frac{2}{\pi L_1 L_2} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\beta(\underline{x})\alpha_{mn}(\underline{x})\beta(\underline{x}_0)\alpha_{mn}(\underline{x}_0)}{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2} \right\}$$

$$\exp\{i\omega(t - t_0)\}\, d\omega \ . \tag{4-205}$$

Note that, owing to the time invariance of this plate system, the Green's function depends only on the time difference between excitation and observation.

With regard to temporal dependence, equations (4-200) and (4-201), and thereby equation (4-205), assume only that the response of the system is such that the temporal transform of the displacement field exists. However, the Green's function must satisfy the temporal constraint that the response cannot anticipate the input. Mathematically, this means

$$\frac{\partial^n g(\underline{x},\underline{x}_0,t - t_0)}{\partial t^n} = 0, \quad \text{for } t < t_0 , \tag{4-206}$$

for all n. By inspection of equation (4-205), the frequency dependence of each term in the summation is of the form

$$S_{mn}(\omega) = \frac{-1}{\mu\left\{\omega^2 - \omega_{mn}^2 - ir\omega/\mu\right\}} , \tag{4-207}$$

where ω_{mn} is the modal natural frequency defined by equation (4-55). Equation (4-207) is of the same mathematical form as equation (3-87). Consequently, by arguments similar to those presented in equations (3-88)-(3-91), it can be shown that the temporal dependence of each term in the summation of equation (4-205) is given by

$$s_{mn}(t) = (1/\mu)U(t - t_0)\exp\{-r(t - t_0)/(2\mu)\} \frac{\sin\{{}_d\omega_{mn}(t - t_0)\}}{{}_d\omega_{mn}} , \tag{4-208}$$

where ${}_d\omega_{mn}$ is the damped modal natural frequency of the simply supported plate, defined by

$${}_d\omega_{mn} = \sqrt{\omega_{mn}^2 - (r/2\mu)^2} . \tag{4-209}$$

It is evident that $s_{mn}(t)$ and all its temporal derivatives are identically zero for $t < t_0$, regardless of the values of m and n. It follows, inasmuch as each term of the summation comprising $g(\underline{x},\underline{x}_0,t - t_0)$ has the temporal dependence specified by equation (4-208), that the Green's function of equation (4-205) satisfies the causal condition of equation (4-206).

By multiplying equation (4-205) by $f(\underline{x}_0,t_0)$ and integrating over all x_0 and t_0, it is evident, by comparing the result with equations (4-200) and (4-201), that

$$w(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,x_0,t - t_0)f(x_0,t_0)\, dx_0\, dt_0 . \tag{4-210}$$

Note that this Green's function solution for the simply supported plate, a system space limited in two dimensions, contains no line integrals representing additional inputs associated with boundary forces. Thus, by the arguments of section 4.3.1.2, the Green's function specified by equation (4-205) is exact.

A note is also in order regarding the forcing function $f(\underline{x},t)$. As used in equations (4-198)-(4-200) and in equation (4-210), $f(\underline{x},t)$ is a function defined over all $\underline{x}$ and t that is equal to the force per unit area applied over the surface of the plate in the spatial range $0 < x_1 < L_1$ and $0 < x_2 < L_2$. Outside this spatial range, $f(\underline{x},t)$ can be arbitrarily specified, inasmuch as forces applied outside the physical extent of the plate do not affect any displacement of the plate.

If we write

$$g(\underline{x},\underline{x}_0,\tau) = (2\pi)^{-5} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\underline{k},\underline{\alpha},\omega)\exp\{\underline{k}\cdot\underline{x} + \underline{\alpha}\cdot\underline{x}_0 + \omega\tau\}\, d\underline{k}\, d\underline{\alpha}\, d\omega , \quad (4\text{-}211)$$

then it follows from equations (4-210) and (4-211) that the wavevector-frequency transform of the displacement field, $W(\underline{k},\omega)$, is related to the wavevector-frequency transform of the forcing field, $F(\underline{k},\omega)$, by

$$W(\underline{k},\omega) = (2\pi)^{-2} \int_{-\infty}^{\infty} G(\underline{k},-\underline{\alpha},\omega)F(\underline{\alpha},\omega)\, d\underline{\alpha} . \quad (4\text{-}212)$$

If we denote the magnitude of the wavevector $\underline{k}_{mn} = (m\pi/L_1, n\pi/L_2)$ associated with the mn-th mode of the plate by

$$k_{mn} = \sqrt{(m\pi/L_1)^2 + (n\pi/L_2)^2} \quad (4\text{-}213)$$

and make use of the definitions of equations (4-62) and (4-70), it is straightforward to show, from equation (4-205), that

$$G(k,\alpha,\omega) = \frac{4}{DL_1L_2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{I_{mn}(k)I_{mn}(\alpha)}{k_{mn}^4 - k_p^4(\omega) + ir\omega/D} . \quad (4\text{-}214)$$

Therefore, by equations (4-212) and (4-214), it follows that

$$W(\underline{k},\omega) = \frac{1}{\pi^2 D L_1 L_2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{I_{mn}(\underline{k})}{k_{mn}^4 - k_p^4(\omega) + ir\omega/D} \right\} \int_{-\infty}^{\infty} I_{mn}^*(\underline{\alpha}) F(\underline{\alpha},\omega)\, d\underline{\alpha} \ . \quad (4\text{-}215)$$

If we make use of the definitions of $I_{mn}(\underline{k})$ and $F(\underline{k},\omega)$, it is straightforward to show that

$$\int_{-\infty}^{\infty} I_{mn}(\underline{\mu}) F(\underline{\mu},\omega)\, d\underline{\mu} = \pi^2 L_1 L_2 B_{mn}(\omega) \ , \quad (4\text{-}216)$$

where $B_{mn}(\omega)$ is the frequency-dependent modal force defined by equation (4-198). Therefore, we can rewrite equation (4-215) as

$$W(\underline{k},\omega) = \frac{1}{D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{B_{mn}(\omega) I_{mn}(\underline{k})}{k_{mn}^4 - k_p^4(\omega) + ir\omega/D} \ . \quad (4\text{-}217)$$

Equations (4-214) and (4-217) (or (4-215)) define, respectively, the two-wavevector-frequency response, $G(\underline{k},\underline{\alpha},\omega)$, and the wavevector-frequency transform, $W(\underline{k},\omega)$, of the displacement field for the forced simply supported plate.

A noteworthy feature of equations (4-214) and (4-217) is that the description of this space-limited field in the wavevector domain does not offer an advantage in mathematical simplicity over the description in the spatial domain. That is, by equations (4-198)-(4-201),

$$w(\underline{x},t) = \frac{1}{2\pi D} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{B_{mn}(\omega)\beta(\underline{x})\alpha_{mn}(\underline{x})}{k_{mn}^4 - k_p^4(\omega) + ir\omega/D} \right\} \exp(i\omega t)\, d\omega \ . \quad (4\text{-}218)$$

Comparison of equation (4-217) with equation (4-218) reveals that, while the transformation from the temporal domain to the frequency domain has resulted in a mathematical simplification by elimination of an integral, both the spatial and wavevector characteristics of the field are expressed as doubly infinite summations of modal functions characteristic of the respective

domains. Similar arguments apply to the Green's function (see equation (4-205)) and its wavevector-frequency transform, $G(\underline{k},\underline{\alpha},\omega)$, defined by equation (4-214). Thus, the prediction of the wavevector characteristics of the space-limited displacement field of a simply supported plate is a mathematical task equally difficult to that of predicting the spatial characteristics.

The mathematical complexity of the expressions for $W(\underline{k},\omega)$ and $G(\underline{k},\underline{\alpha},\omega)$ preclude us from attempting any detailed analysis of the wavevector-frequency characteristics of the forced motion of the simply supported plate. However, by examination of equations (4-214) and (4-217), we can identify significant contributions to $W(\underline{k},\omega)$ and $G(\underline{k},\underline{\alpha},\omega)$ and thereby gain insight into the general nature of these wavevector-frequency descriptions.

We first note, from equation (4-212), that $G(\underline{k},\underline{\alpha},\omega)$ defines the complex amplitudes of waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ that comprise the displacement field of the plate as a result of excitation of the plate by the wave $(2\pi)^{-1} \exp\{i(-\underline{\alpha}\cdot\underline{x} + \omega t)\}$. Similarly, equation (4-215) or (4-217) relates the complex amplitudes, $W(\underline{k},\omega)$, of the waves of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ that comprise the displacement field of the plate to, respectively, the complex amplitudes, $F(\underline{\alpha},\omega)$, of the waves of the form $\exp\{i(\underline{\alpha}\cdot\underline{x} + \omega t)\}$ or the frequency-dependent modal forces, $B_{mn}(\omega)$, that comprise the forcing field of the plate.

By inspection of equations (4-214) and (4-217), it is evident that, at any fixed frequency, ω_0, the dependence of both $G(\underline{k},\underline{\alpha},\omega_0)$ and $W(\underline{k},\omega_0)$ on the wavevector $\underline{k}$ is specified by a weighted superposition of wavevector transforms of the space-limited natural modes, $I_{mn}(\underline{k})$, over all mode numbers, m and n. Two separate functions weight $I_{mn}(\underline{k})$ at each value of m and n. One weighting function, the term

$$\left\{k_{mn}{}^4 - k_p{}^4(\omega_0) + ir\omega_0/D\right\}^{-1} ,$$

specifies the response of the mn-th mode of the plate at the frequency ω_0. The other weighting function is the frequency dependent modal force, $B_{mn}(\omega_0)$, acting on the plate. In equation (4-214), this modal force is frequency independent and is given by $I_{mn}(\underline{\alpha})$, where $-\underline{\alpha}$ is the fixed, but

arbitrary, wavevector characterizing the (single) complex wave that forces the motion of the plate. It is important to note that $I_{mn}(\underline{k})$ and the two weighting functions are, in general, complex.

Clearly, the relation between the wavevector-frequency characteristics of the displacement field and those of the forcing field is not a mathematically simple one. However, we can gain some insight into the wavevector-frequency characteristics of the displacement field by identifying those terms in the summations of equations (4-124) and (4-217) that provide the largest, in magnitude, wavevector contributions to the displacement field at some fixed, but arbitrary, frequency ω_0. We will perform this identification for $W(\underline{k},\omega)$, described by equation (4-217). The general wavevector-frequency characteristics of $G(\underline{k},\underline{\alpha},\omega)$ can then be examined as a special case of $W(\underline{k},\omega)$.

From our investigation of the free vibration of the simply supported plate (section 4.2.2), we know that the magnitude of $I_{mn}(\underline{k})$ has four equal primary maxima at the wavevectors $\underline{k} = (\pm m\pi/L, \pm n\pi/L_2)$ and has secondary maxima at the wavevectors $\underline{k} = \{\pm[m \pm (2p + 1)]\pi^2/L_1, \pm[n \pm (2q + 1)]\pi/L_2\}$ for all integers p and q equal to or greater than one. The magnitudes of these secondary maxima decrease with increasing wavevector distance, $\underline{\varepsilon}$, from the primary maxima approximately as $(|\varepsilon_1|\,|\varepsilon_2|)^{-1}$ and are therefore considerably smaller than the magnitudes at $\underline{k} = (\pm m\pi/L_1, \pm n\pi/L_2)$. The reader can refresh his memory regarding the wavevector characteristics of $|I_{mn}(\underline{k})|$ by referring to figure 4-4. It follows, by the above arguments, that the largest, in magnitude, wavevector contributions to $W(\underline{k},\omega_0)$ from each mn-th term in the summation of equation (4-215) occur at the four wavevectors $\underline{k} = (\pm m\pi/L_1, \pm n\pi/L_2)$. These wavevectors associated with the primary maxima of $I_{mn}(\underline{k})$ are referred to as the modal wavevectors, and their magnitudes, k_{mn}, as modal wavenumbers (see equation (4-213)).

By equation (4-64), it can be shown that the magnitudes of the four primary maxima of $I_{mn}(k)$ are equal and independent of the mode numbers m and n. Therefore, from equation (4-217), it is evident that (1) the largest, in magnitude, wavevector contributions to $W(\underline{k},\omega_0)$ from each mode occur at the wavevectors $\underline{k} = (\pm m\pi/L_1, \pm n\pi/L_2)$ and (2) the magnitudes of the contributions at each of these modal wavevectors are determined by the magnitudes of the

weightings applied to these modes: that is, by the magnitude of the product of $B_{mn}(\omega_0)$ and $\{k_{mn}^4 - k_p^4(\omega_0) + ir\omega_0/D\}^{-1}$. In the absence of specific knowledge of the modal forces, $B_{mn}(\omega_0)$, the complete set of modal wavevectors must be considered as sites of potentially large contributions to $W(\underline{k},\omega_0)$. Figure 4-7 illustrates the wavevector locations of this set of modal wavevectors.

Also illustrated in figure 4-7 is a circle of radius equal to the free wavenumber of the plate, $k_p(\omega_0)$, at the frequency ω_0. A coincidence of the magnitude of a modal wavevector, say k_{MN}, with $k_p(\omega_0)$ defines a resonance of the MN-th mode plate. By equation (4-71), this resonance occurs at the frequency $\omega_0 = \omega_{MN}$. At resonance, the magnitude of the weighting function that specifies the response of the mn-th mode of the plate at the frequency ω_0, i.e., $\{k_{mn}^4 - k_p^4(\omega_0) + ir\omega_0/D\}^{-1}$, reaches its maximum value. That is, at the frequency ω_{MN} where the MN-th mode is resonant, the magnitude of the MN-th modal contribution to $W(\underline{k},\omega_{MN})$ is weighted by

$$\frac{|B_{MN}(\omega_{MN})|}{|\{k_{MN}^4 - k_p^4(\omega_{MN}) + ir\omega_{MN}/D\}|} = \frac{|B_{MN}(\omega_{MN})|}{|(r\omega_{MN}/D)|} . \tag{4-219}$$

By contrast, the magnitude of a nonresonant (say PQ-th) modal contribution to $W(\underline{k},\omega_{MN})$ is weighted by

$$\frac{|B_{PQ}(\omega_{MN})|}{|\{k_{PQ}^4 - k_p^4(\omega_{MN}) + ir\omega_{MN}/D\}|} .$$

Clearly, for equal magnitudes of excitation of the resonant MN-th and the nonresonant PQ-th modes (i.e., $|B_{MN}(\omega_{MN})| = |B_{PQ}(\omega_{MN})|$), the ratio of the magnitude of the weighting applied to the MN-th modal contribution to $W(\underline{k},\omega_{MN})$ to that applied to the PQ-th contribution is

$$\frac{|\{k_{PQ}^4 - k_p^4(\omega_{MN}) + ir\omega_{MN}/D\}|}{|(r\omega_{MN}/D)|} > 1 \tag{4-220}$$

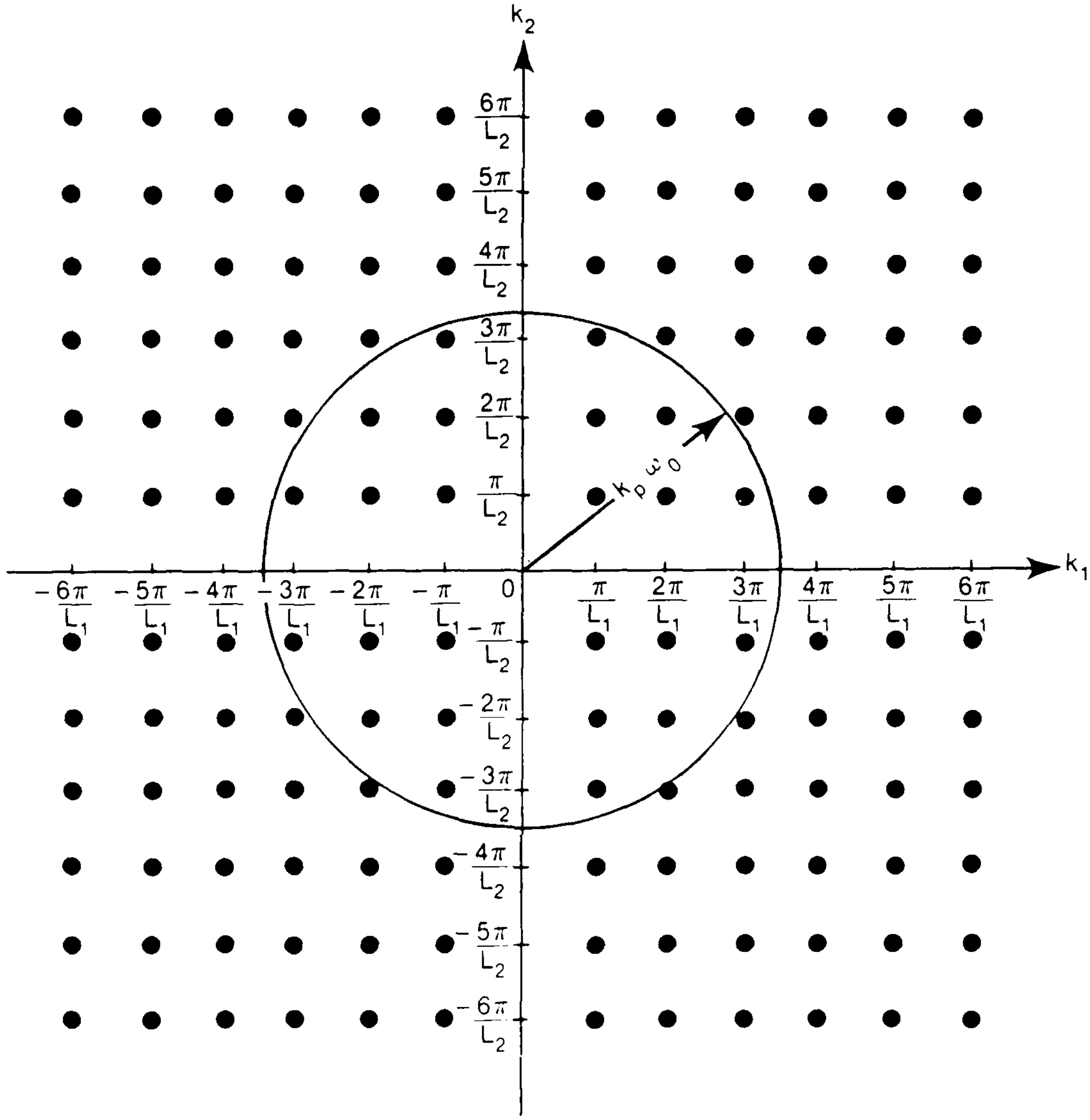

Figure 4-7. Wavevector Locations of Potentially Large Contributions to the Magnitude of $W(\underline{k},\omega)$

Thus, in figure 4-7, the modes characterized by modal wavevectors on or near the circle defining the free wavenumber of the plate (i.e., those where $k_{mn} \approx k_p(\omega_{MN})$) respond more strongly to modal forces than do those modes characterized by modal wavevectors well inside or outside that circle. By equation (4-220), the magnitude of the relative response of resonant to nonresonant modes depends on both the damping and the relative magnitudes of the modal and free wavenumbers.

Let us now examine how knowledge of the magnitudes of the maximum wavevector contributions from the various terms in the summation of equation (4-217) can be used to gain insight into the wavevector characteristics of $W(\underline{k},\omega)$ at a fixed, but arbitrary, frequency, ω_0. We assume that the plate properties and dimensions are known and, therefore, that the modal natural frequencies, modal wavevectors, and the damping are known quantities. We also assume that the modal forces are known. We know, from basic vibration theory, that the response of the plate will be greatest at the modal natural frequencies of the plate. Therefore, the characteristics of $W(\underline{k},\omega)$ at these natural frequencies are of primary interest. Let us therefore examine the characteristics of $W(\underline{k},\omega_{MN})$, where ω_{MN} is the natural frequency of the MN-th mode of the plate.

By equation (4-71) and (4-217),

$$W(\underline{k},\omega_{MN}) = \frac{1}{D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}(\omega_{MN})I_{mn}(\underline{k})}{k_{mn}^{4} - k_{MN}^{4} + ir\omega_{MN}/D} . \tag{4-221}$$

By our previous arguments, we know that the maximum, in magnitude, wavevector contributions to $W(\underline{k},\omega_{MN})$ from each term occur at the mn-th modal wavevectors, $\underline{k} = (\pm m\pi/L_1, \pm n\pi/L_2)$. Further, we know that $\{k_{mn}^{4} - k_{MN}^{4} + ir\omega_{MN}/D\}^{-1}$ is a maximum when $k_{mn} = k_{MN}$, or when $m = M$ and $n = N$. Thus, ignoring the relative magnitudes of the modal forces, we would expect significant contributions to $W(\underline{k},\omega_{MN})$ at the modal wavevectors $\underline{k} = (\pm M\pi/L_1, \pm N\pi/L_2)$. On the other hand, if we assume that one modal force, say B_{PQ}, is much larger than the others, we might also expect significant contributions to $W(\underline{k},\omega_{MN})$ at the modal wavevectors $\underline{k} = (\pm P\pi/L_1, \pm Q\pi/L_2)$. Let us first look at the relative magnitudes of the modal contributions to $W(\underline{k},\omega_{MN})$ at the wavevector $\underline{k} = (M\pi/L_1, N\pi/L_2)$.

The magnitude of the contribution from the MN-th modal term is

$$\frac{|B_{MN}(\omega_{MN})|\ \ |I_{MN}(M\pi/L_1, N\pi/L_2)|}{|r\omega_{MN}/D|} ,$$

and the magnitudes of the contributions from the nonreasonant terms are

$$\frac{|B_{mn}(\omega_{MN})| \; |I_{mn}(M\pi/L_1, N\pi/L_2)|}{|k_{mn}^4 - k_{MN}^4 + ir\omega_{MN}/D|} .$$

The largest nonresonant contributions will be those from modes adjacent to the resonant mode and from the mode with the large modal force, the PQ-th mode. If we assume that the closest nonresonant mode is the M-1, N-th mode, it can be established from equations (4-63) and (4-64) that

$$|I_{M-1,N}(M\pi/L_1, N\pi/L_2)| \approx (2/\pi)\,|I_{MN}(M\pi/L_1, N\pi/L_2)| . \tag{4-222}$$

Therefore, the ratio of the magnitude of the resonant modal contribution to that of an adjacent, nonresonant mode is given by

$$\frac{\pi |B_{MN}(\omega_{MN})| \; |k_{M-1,N}^4 - k_{MN}^4 + ir\omega_{MN}/D|}{2 |B_{M-1,N}(\omega_{MN})| \; |r\omega_{MN}/D|} .$$

For modal forces of comparable magnitudes, large wavevector separations between adjacent modes, and small damping, this ratio can be large (say of the order of 10). Conversely, for small modal separations, large damping, and comparable modal forces, this ratio is just slightly greater than 1. If the M-1,N-th mode were subjected to a much larger modal force than the MN-th mode, this ratio could be less than 1.

Consider now the magnitude of the contribution from the PQ-th mode (where the modal force is significantly larger than other modal forces) to $W(M\pi/L_1, N\pi/L_2, \omega_{MN})$. Because we have already looked at the contributions of adjacent modes, we will assume that the PQ-th mode is somewhat removed from the resonant MN-th mode. The magnitude of the contribution from the PQ-th mode to $W(M\pi/L_1, N\pi/L_2, \omega_{MN})$ is given by

$$\frac{|B_{PQ}(\omega_{MN})| \; |I_{PQ}(M\pi/L_1, N\pi/L_2)|}{|k_{PQ}^4 - k_{MN}^4 + ir\omega_{MN}/D|} .$$

However, for $|M-P|$ and $|N-Q|$ greater than 2, we can show that

$$|I_{PQ}(M\pi/L_1, N\pi/L_2)| \le \frac{|I_{PQ}(P\pi/L_1, Q\pi/L_2)|}{|M - P| \; |N - Q|} = \frac{|I_{MN}(M\pi/L_1, N\pi/L_2)|}{|M - P| \; |N - Q|} .$$

Thus, the ratio of the magnitude of the resonant MN-th modal contribution to $W(M\pi/L_1, N\pi/L_2, \omega_{MN})$ to the magnitude of the PQ-th modal contribution can be shown to be equal to or greater than

$$\frac{|B_{MN}(\omega_{MN})|\ |M - P|\ |N - Q|\ |k_{PQ}^4 - k_{MN}^4 + ir\omega_{MN}/D|}{|B_{PQ}(\omega_{MN})|\ |r\omega_{MN}/D|} .$$

Because we have assumed that $|M-P|$ and $|N-Q|$ are both greater than 2, it follows that, when the modal separations are sufficiently large and the damping sufficiently small that

$$\frac{|k_{PQ}^4 - k_{MN}^4 + ir\omega_{MN}/D|}{|r\omega_{MN}/D|} \approx 10 ,$$

the magnitude of the PQ-th modal force must be nearly two orders of magnitude greater than that of the MN-th modal force in order for the magnitude of the PQ-th modal contribution to be of the order of the magnitude of the MN-th modal contribution.

By the above arguments, it follows that, for sufficiently large modal separations, sufficiently small damping, and modal forces that exhibit large variations in magnitude only at modes well removed from resonance, the magnitude of the resonant MN-th modal contribution to $W(M\pi/L_1, N\pi/L_2, \omega_{MN})$ is at least an order of magnitude greater than the magnitudes of each of the other modal contributions. It is also easily verified that the phases of the various modal contributions vary with the mode numbers m and n. Therefore, if we envision each modal contribution as a vector, it is reasonable to argue that the sum of one large vector (the MN-th contribution) with many small vectors of random direction (the nonresonant modal contributions) results in a vector nearly equal to the original large vector. By this argument, it follows that, under the above restrictions,

$$|W(M\pi/L_1, N\pi/L_2, \omega_{MN})| \approx \frac{|B_{MN}(\omega_{MN})|\ |I_{MN}(M\pi/L_1, N\pi/L_2)|}{D\,|r\omega_{MN}/D|} . \qquad (4\text{-}223)$$

This estimate ; also valid at the other three wavevectors associated with the MN-th mode: tl t is, at $\underline{k} = (-M\pi/L_1, N\pi/L_2)$, $\underline{k} = (M\pi/L_1, -N\pi/L_2)$, and $\underline{k} = (-M\pi/L_1, -\!\!\!\quad/L_2)$.

As we rel the restrictions on modal separation, damping, and variations in modal forc , the contributions from some of the nonresonant modes increase, and ie estimate of the magnitude of $W(\underline{k},\omega_{MN})$ at the resonance wavevectors b the maximum magnitude of the MN-th modal term becomes a poorer one. Inasmuc is the magnitude of a sum is less than or equal to the sum of the magnitude the estimate of the magnitude of $W(M\pi/L_1, N\pi/L_2, \omega_{MN})$ under these r ixed conditions will likely decrease from its true value. However, for isonable relaxations of these conditions, it is likely that $W(\underline{k},\omega_{MN})$ will :ill exhibit relative maxima in the neighborhoods of $\underline{k} = (\pm M\pi/L_1, \pm\quad/L_2)$.

Arguments milar to the above may be applied to estimate the magnitude of $W(k,\omega_{MN})$ at t modal wavevector $\underline{k} = (P\pi/L_1, Q\pi/L_2)$ associated with the modal force t . was assumed large in comparison with the others. By such arguments, it in be demonstrated that, at $\underline{k} = (P\pi/L_1, Q\pi/L_2)$ and $\omega = \omega_{MN}$, the ratio of the gnitude of the PQ-th modal contribution to that of the MN-th contribution given by

$$\frac{|B_{PQ}(\omega_{MN})|\;|M - P|\;|N - Q|\;|r\omega_{MN}/D|}{|B_{MN}(\omega_{MN})|\;|k_{PQ}{}^4 - k_{MN}{}^4 + ir\omega_{MN}/D|} .$$

If we again a ime values of damping and modal wavevector separations such that

$$\frac{|k_{PQ}{}^4 - k_{MN}{}^4 + ir\omega_{MN}/D|}{|r\omega_{MN}/D|} \approx 10 ,$$

then, because have assumed that $|M-P|$ and $|N-Q|$ are both greater than 2, it follows that ratic of the magnitude of the PQ-th contribution to that of the MN th con ibution is of the order $|B_{PQ}(\omega_{MN})|/|B_{MN}(\omega_{MN})|$. Thus, if $|B_{PQ}(\omega_{MN})|$ an order of magnitude larger than $|B_{MN}(\omega_{MN})|$, then, by the argume presented previously, the magnitude of $W(P\pi/L_1, Q\pi/L_2, \omega_{MN})$

can be approximated by

$$|W(P\pi/L_1, Q\pi/L_2, \omega_{MN})| \approx \frac{|I_{PQ}(P\pi/L_1, Q\pi/L_2)|\,|B_{PQ}(\omega_{MN})|}{D|k_{PQ}{}^4 - k_{MN}{}^4 + ir\omega_{MN}/D|} \quad . \tag{4-224}$$

It is interesting to compare this estimate with the estimate of the magnitude of $W(\pm M\pi/L_1, \pm N\pi/L_2, \omega_{MN})$ obtained previously under similar assumptions of damping, modal separations, and relative magnitudes of modal forces. By equations (4-223) and (4-224), we can show

$$\frac{|W(M\pi/L_1, N\pi/L_2, \omega_{MN})|}{|W(P\pi/L_1, Q\pi/L_2, \omega_{MN})|} \approx \frac{|B_{MN}(\omega_{MN})|\,|k_{PQ}{}^4 - k_{MN}{}^4 + ir\omega_{MN}/D|}{|B_{PQ}(\omega_{MN})|\,|\omega_{MN}/D|} \quad . \tag{4-225}$$

For the small damping, large modal separations and relative magnitudes of modal forces used to obtain equations (4-223) and (4-224), this ratio is of the order of unity. Thus, we can conclude that the magnitude of $W(\underline{k},\omega)$ at the resonance frequency ω_{MN} is characterized by equally large contributions at the four wavevectors $\underline{k} = (\pm M\pi/L_1, \pm N\pi/L_2)$ associated with the resonance of the MN-th mode and at the four wavevectors $\underline{k} = (\pm P\pi/L_1, \pm Q\pi/L_2)$ associated with the modal response of the strongly driven PQ-th mode.

The reader should be reminded once again that the above approximations, and thereby the above conclusions, are strictly valid only for small damping, large modal separations, and the ratios of modal force magnitudes used for their derivations. However, it is likely that, under reasonable relaxations of these restrictions, the magnitude of $W(\underline{k},\omega)$ at a resonance frequency ω_{MN} will be relatively large in the neighborhoods of those modal wavevectors associated with resonances and in the neighborhoods of those modal wavevectors associated with relatively large modal forces. Indeed, it seems reasonable to expect that, at any frequency, the magnitude of the $W(\underline{k},\omega)$ will be relatively large in the neighborhoods of the (four) modal wavevectors associated with each resonant or near-resonant mode and in the neighborhoods of the modal wavevectors associated with relatively large modal forces. The exact wavevector locations of these relative maxima of $|W(\underline{k},\omega)|$ will depend on the relative separation between the modal wavevectors associated with the

resonant and near-resonant modes and on the exact distribution of modal forces. To the degree that such an extension of the highly specialized example presented above is a valid one, the plot of the modal wavevectors and the radius of the free wavenumber shown in figure 4-7 can be a useful tool for identifying potentially large wavevector contributions to $W(\underline{k},\omega)$.

As stated previously, the two-wavevector-frequency response, $G(\underline{k},\underline{\alpha},\omega)$, can be treated as a special case of the wavevector-frequency description of the forced displacement field, $W(\underline{k},\omega)$, of the simply supported plate. That is, $G(\underline{k},\underline{\alpha},\omega)$ is the wavevector-frequency transform of the displacement field when the space-time forcing field is the single complex wave given by $(1/2\pi)\exp\{-i(\underline{\alpha}\cdot\underline{x} - \omega t)\}$. As can be seen, by comparison of equations (4-214) and (4-217), the modal forces associated with this forcing field are $I_{mn}(\underline{\alpha})$, independent of the frequency characterizing the forcing wave.

By applying arguments similar to those used to investigate the wavevector behavior of $W(\underline{k},\omega)$, the most significant wavevector-frequency characteristics of $G(\underline{k},\underline{\alpha},\omega)$ can be deduced. Inasmuch as the procedure used to identify these characteristics is identical to that employed previously, we will omit the details of this deduction process. However, the reader is encouraged to perform such a detailed analysis to gain familiarity and confidence with this predictive technique.

Consider first the response of the plate to the wave $\exp\{-i(P\pi x_1/L_1 + Q\pi x_2/L_2 - \omega_{MN}t)\}$; thus, $\underline{\alpha} = (P\pi/L_1, Q\pi/L_2)$, one of the four modal wavevectors of the PQ-th mode of the plate, and $\omega = \omega_{MN}$, the natural frequency of the MN-th mode. It is straightforward to show that for small damping, large modal separations, and $|P-M|$ and $|Q-N|$ greater than 2, the largest contributions to $|G(\underline{k};P\pi/L_1,Q\pi/L_2;\omega_{MN})|$ occur at the four wavevectors, $\underline{k} = (\pm M\pi/L_1, \pm N\pi/L_2)$, associated with the resonant MN-th mode and at the four wavevectors, $\underline{k} = (\pm P\pi/L_1, \pm Q\pi/L_2)$, associated with the mode that includes the wavevector of excitation. The relative magnitudes of the contributions at the resonant modal wavevectors and at the modal wavevectors associated with the input depend on the exact values of the damping, the modal separation, and the differences $|P-M|$ and $|Q-N|$. It should be recognized that $G(\underline{k},\omega)$ is zero in this example when $\omega \neq \omega_{MN}$.

It is easily shown that if the plate is forced at resonance, i.e., by $\exp\{-i(M\pi x_1/L_1 + N\pi x_2/L_2 - \omega_{MN}t)\}$, $|G(\underline{k};M\pi/L_1,N\pi/L_2;\omega_{MN})|$ is characterized by only four large contributions: one at each of the modal wavevectors $\underline{k} = (\pm M\pi/L_1, \pm N\pi/L_2)$.

By extending these results to arbitrary wavevectors and frequencies of excitation, it appears that the magnitude of $G(\underline{k},\underline{\alpha},\omega)$, at the frequency of excitation, will be relatively large in the neighborhoods of the four modal wavevectors associated with each resonant or near-resonant mode and in the neighborhoods of those modal wavevectors that have a member close in amplitude and direction to the wavevector characterizing the wave input to the plate. The exact wavevector locations of these relative maxima of $|G(\underline{k},\underline{\alpha},\omega)|$ will depend on the relative separation between the modal wavevectors associated with the resonant and near-resonant modes and on the wavevector characterizing the single wave excitation of the plate.

By the arguments presented above, we have demonstrated two important features of the wavevector-frequency response of the space-limited plate. The first is that the plate responds most strongly, at any given frequency, to those wave components of the input field that are characterized by wavevectors closest, in magnitude, to the free wavenumber of the plate. The second is that any single wavevector-frequency component of excitation to the plate produces, at the frequency of excitation, a strong response at not only those modal wavevectors with members most nearly coinciding with the wavevector of excitation, but also at those modal wavevectors close, in magnitude, to the free wavenumber of the plate. These resonant components of the plate motion are a consequence of the reflections (or wavevector scattering) of the forced waves at the boundaries of the plate.

4.3.3.3 Observations From Illustrative Examples. The two relatively simple examples of space-limited systems presented above provide ample evidence that the analysis and interpretation of the wavevector-frequency characteristics of space-limited systems is a considerably more complex task than that of analyzing and interpreting space-invariant systems in the wavevector-frequency domain.

By use of these examples, we have shown that the wavevector-frequency response of space-limited systems has two characteristics in common with the wavevector-frequency response of space-invariant systems. The first characteristic is that both space-limited and space-invariant systems respond most strongly, at any frequency, to wavevector components of excitation equal, in magnitude, to the free wavevector of the system. The second common characteristic is that the magnitudes of the wavevector-frequency transform of the output field of both space-limited and space-invariant systems exhibit, at any given frequency, relative maxima in the neighborhoods of those wavevectors characterizing relatively large inputs to the system.

However, the example of the simply supported plate illustrates a characteristic of the wavevector-frequency response of space-limited systems not encountered in space-invariant systems. That characteristic is wavevector scattering (or conversion). Recall that in space- (and time-) invariant systems, a single wavevector-frequency component of input produces only a single wavevector-frequency component of response, with that response component occurring at the wavevector and frequency characterizing the input. In the space-limited system of the simply supported plate, we found that a single wavevector-frequency component of input produced, at any frequency, a continuum of wavevector components in the output as a result of reflection, or scattering, of the input wave from the boundaries of the plate. The largest (in magnitude) components in the output field occurred at the (four) modal wavevectors associated with each resonant (or near-resonant) mode at that frequency and at the modal wavevectors with members nearly equal to the wavevector of excitation. The most important feature of this wavevector scattering is the excitation of wavevector components close, in magnitude, to that of the free wavenumber of the plate by inputs characterized by wavevectors far removed, in their magnitudes, from the free wavenumber.

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CHAPTER 5
COUPLED LINEAR SYSTEMS

The spatially distributed, time-invariant linear systems treated in the previous two chapters have consisted of a single physical component, such as a string, a plate, or an acoustic fluid. However, in such specialized fields as structural acoustics, musical acoustics, architectural acoustics, and noise, the systems of practical interest are comprised of multiple interacting physical components. This chapter addresses the response of spatially distributed, time-invariant linear systems comprised of more than one physical component.

5.1 FUNDAMENTAL CONCEPTS OF COUPLED SYSTEMS

In section 3.1 of chapter 3, we defined a system as "an aggregation or assemblage of interacting elements combined by man or nature to form an integral entity." In a spatially distributed system comprised of a single physical component, the "interacting elements" of the system are differential lengths, areas, or volumes (as appropriate) of the physical component under scrutiny. For a spatially distributed system comprised of multiple interacting physical components, the "interacting elements" of the composite system are the individual physical components of the system. Thus, we see that for a spatially distributed single component system, the elements of the system are of microscopic spatial scale, whereas for a spatially distributed multicomponent system, the elements of the composite system are of macroscopic scale. This difference in spatial scale, though initially somewhat confusing, is simply a consequence of defining a system in terms of "interacting elements."

The macroscopic spatial scale of the elements of a system comprised of multiple physical components does present a dilemma. It has been demonstrated, by the illustrative examples of the preceding chapters, that the spatial scale of the interacting elements of a system corresponds to the

spatial detail to which the system and its response can be described. Further, by definition, the wavevector-frequency description of the response of a system requires knowledge of the system response over all space and time. Clearly then, the macroscopic spatial scales associated with the various physical components of a multicomponent system are not compatible with a wavevector-frequency description of that system.

The solution to this dilemma is really quite simple. We define each physical component (or interacting element) of the multicomponent system to be a subsystem. Each subsystem is now a single component system and is comprised of elements having spatial scales of differential order. The various subsystems are then coupled in accordance with the interactions between the various physical components (or macroscopic elements) of the composite system.

By these arguments, a spatially distributed system comprised of several (say N) interacting physical components can be interpreted as an assemblage of N-coupled subsystems, where each subsystem represents a single physical component of the composite system, and the couplings between the subsystems are chosen to reflect the appropriate interactions between the physical components of the composite system. The title of this chapter, Coupled Linear Systems, was chosen to reflect this interpretation of, and approach to, spatially distributed, time-invariant multicomponent linear systems.

5.1.1 The Causes and Effects of Coupling

Consider a system comprised of several interacting physical components. For an interaction to exist between any two of the various physical components, those components must be either in physical contact, physically connected through one or more of the other components of the system, or subject to some physical field that induces mutual forces between the two components. In the interpretation of a multicomponent system as an assemblage of coupled single component subsystems, the interactions between the various physical components of the composite system define the couplings between the various subsystems. The specific nature of the coupling between any two subsystems can be determined only by examining the physics of the particular interaction between the corresponding components of the composite system.

However, certain characteristics of the coupling between subsystems and, in some cases, of the physical subsystems themselves can be deduced from the specific form of the interaction between them.

Consider the situation in which the interaction between two components of the composite system results from physical contact between the components. The line or surface of physical contact between the two components establishes a spatial boundary common to both components. It therefore follows that the coupling between the physical subsystems corresponding to these components occurs at the boundaries of the subsystems. It further follows that the physical subsystems corresponding to these components are space limited.

Arguments similar to the above apply when the interaction between two components of a composite system results from an interconnection of these components via one or more of the other components of the composite system. That is, the lines or surfaces of physical contact between the components of interest and the interconnecting component(s) establish spatial boundaries for both components of interest, thereby spatially limiting the corresponding physical subsystems. In addition, the coupling between these physical subsystems acts at these boundaries.

Two components of a composite system can interact through the presence of some field of physical origin (an electromagnetic field, for example) between the components. Such fields can produce interactive forces between components at the atomic or molecular level in the absence of any physical contact between the components. It follows that, in the presence of such interactions between components, the coupling between the associated physical subsystems acts not only at the boundaries of the subsystems, but can act between any of the (differential scale) elements of the two subsystems. All physical subsystems associated with a multicomponent system are, of course, space limited inasmuch as two components cannot simultaneously occupy all of space.

5.1.2 Classification of Coupled Systems

From the above discussion, it would appear that, inasmuch as each of the various physical subsystems associated with a given multicomponent system is

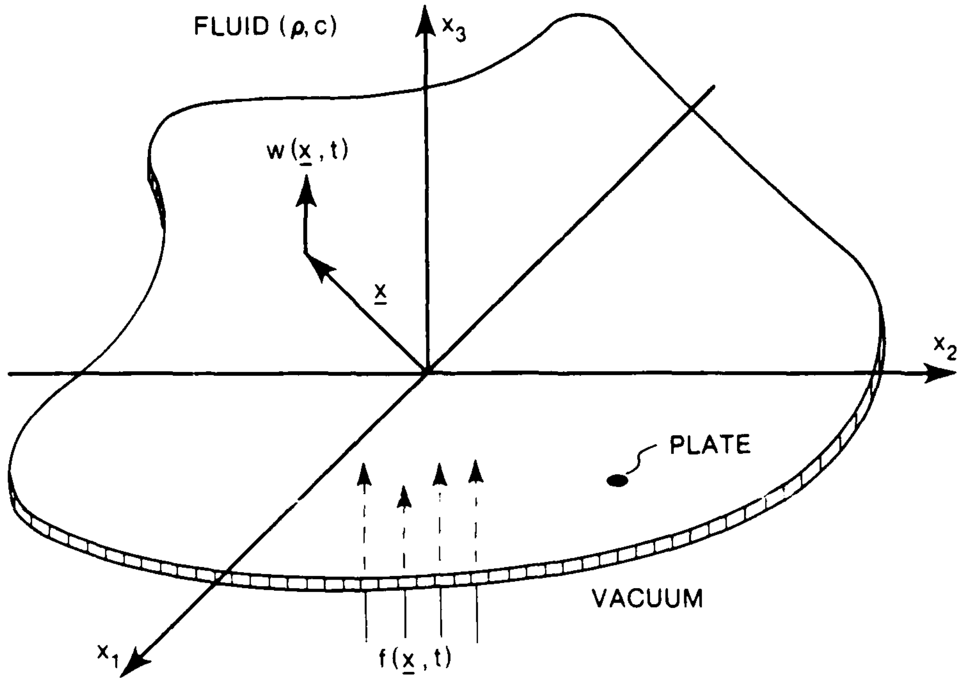

Figure 5-1. Geometry of Fluid-Loaded Plate

plate and fluid are in contact at the top surface of the plate, the displacement field of the plate is imposed on the fluid at this boundary, thereby exciting a pressure field, $p(\underline{x},x_3,t)$, in the acoustic half space. This pressure field, acting on the top surface of the plate, produces an additional input field, $p(\underline{x},0,t)$, to the plate that acts opposite in direction to the external input, $f(\underline{x},t)$. We have assumed here that, with respect to the acoustic half space, the plate is of infinitesimal thickness. Further, we have applied the law of conservation of mass (or continuity), which requires that the displacement field of the plate and the component of the displacement field of the fluid normal to the plane of the plate be equal at the plate-fluid interface ($x_3 = 0$).

By means of the above arguments and figure 5-2, we can completely specify the subsystems associated with the infinite plate and the acoustic half-space components of the composite plate-fluid system and the coupling between them. The elements of the infinite plate subsystem are subjected to two inputs: the externally imposed input, $f(\underline{x},t)$, and the oppositely directed pressure field, $p(x,0,t)$, that is induced in the acoustic half space and acts on the upper

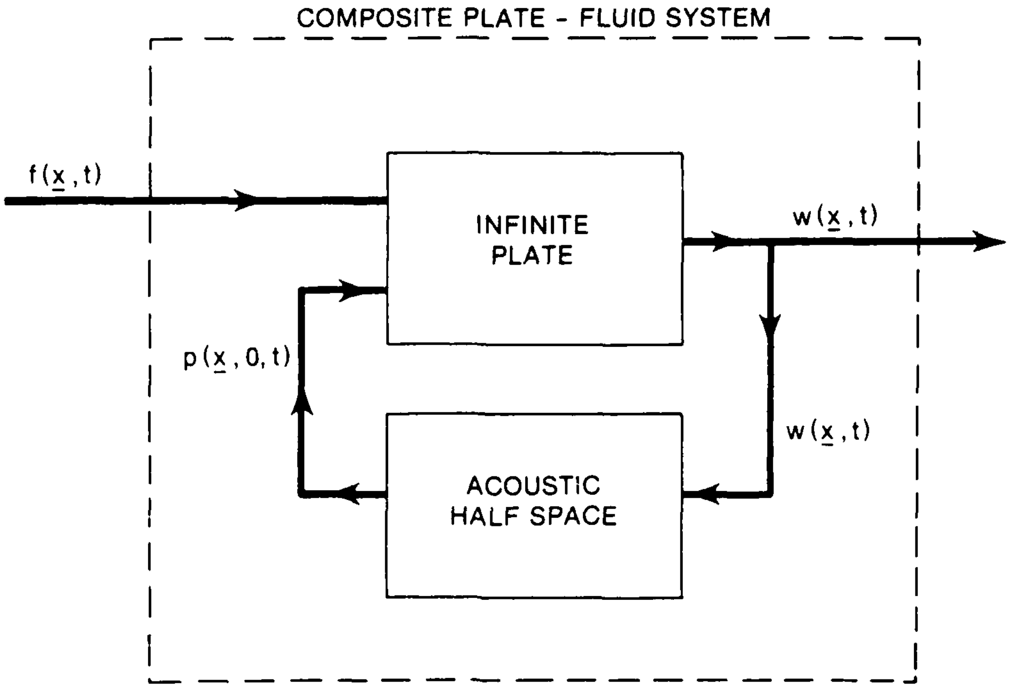

Figure 5-2. Schematic Diagram of the Fluid-Loaded Infinite Plate System

surface of the plate. The only output of consequence from the infinite plate is the vibratory displacement field of the plate, $w(\underline{x},t)$. With the exception of the additional input of the pressure field from the half space, the infinite plate subsystem is identical to the infinite plate system treated in chapter 3 as an illustrative example of a forced, linear, space- and time-invariant system.

Contrary to the impression conveyed by figure 5-2, no inputs (or sources) are applied to all elements of the acoustic half space. Rather, the connection of the output of the infinite plate subsystem to the acoustic half-space subsystem in figure 5-2 indicates that the half-space subsystem is coupled to the infinite plate subsystem via the displacement field of the plate. In this coupling, the displacement of each element of the plate (measured normal to the plane of the plate) is imposed on the contacting element of the acoustic half space over the planar boundary of the half space, $x_3 = 0$, prescribed by the plate. The pressure field in the half space results solely from this imposition of the displacement field of the plate on the boundary of the half space. The quantity of interest, or output, from the

acoustic half-space subsystem is the acoustic pressure field over the planar surface, $x_3 = 0$, that defines the plate-fluid interface. The subsystem associated with the acoustic fluid component of the composite plate-fluid system is a special case of the linear, time-invariant, space-limited acoustic half-space system treated in section 4.3.3.1 of chapter 4.

In identifying the subsystem associated with the acoustic fluid as a special case of the space-limited acoustic half-space system treated in section 4.3.3.1, the words "special case" must be emphasized. The system treated in section 4.3.3.1 was that of the pressure field produced in the acoustic half-space $x_3 \geq 0$ as a result of a prescribed displacement field applied to the boundary of the half space at $x_3 = 0$. This system was shown to be space invariant in the x_1 and x_2 coordinate directions but, owing to the boundary at $x_3 = 0$, space limited in the x_3 coordinate direction. Consequently, this acoustic half-space system was classified as a space-limited system. The subsystem associated with the acoustic fluid component of the composite plate-fluid system is a special case of the half-space system treated in section 4.3.3.1, because the output of this subsystem is the pressure field over the two-dimensional surface defined by $x_3 = 0$, a subset of the three-dimensional half-space $x_3 \geq 0$. However, inasmuch as the pressure field over the plane $x_3 = 0$ (or, for that matter, any plane of constant x_3) is independent of x_3, it is space invariant. Consequently, as can be demonstrated by comparison of the mathematical forms of the input-output relationships of equations (4-159) (at $x_3 = 0$) and (3-59), the subsystem associated with the acoustic fluid component of the composite plate-fluid system is space invariant.

By the use of figure 5-2 and the above arguments, we have defined the coupled plate and acoustic fluid subsystems that will be used to mathematically model the composite system of the fluid-loaded plate. Both subsystems have been argued to be linear, time invariant, and space invariant. It therefore follows that the mathematical model of the composite plate-fluid system is also linear, time invariant, and space invariant.

By reference to figure 5-2, it is evident that the plate and acoustic fluid subsystems are mutually coupled. That is, the acoustic fluid subsystem

is coupled to the plate subsystem through the common displacement field at the interface of these two systems. In addition, the plate subsystem is coupled to the acoustic fluid subsystem by means of the pressure field exerted, by the fluid, on the upper surface of the plate. Because of this mutual coupling, the composite plate-fluid system is a form of feedback system. That is, the output of the composite system (the displacement field of the plate) is fed back through the acoustic fluid subsystem to produce a pressure field that augments the input to the composite system, $f(\underline{x},t)$, in exciting the infinite plate subsystem.

Recall (see figure 5-2) that the input to the composite plate-fluid system described above is the force per unit area, $f(\underline{x},t)$, applied to the infinite plate subsystem and the output was defined to be the displacement field of the plate, $w(\underline{x},t)$. If we redefine the output of the composite system to be the pressure field, $p(\underline{x},x_3,t)$, throughout the acoustic half-space $x_3 \geq 0$ rather than the displacement field of the plate, we specify a new composite system that has the same physical elements (or components) and the same interactions between components as the composite system used to predict the displacement field. The redefinition of the output, therefore, has no effect on the definition of the interacting elements of the two subsystems associated with the plate and acoustic fluid components of the composite system.

To complete the specification of the coupled subsystems associated with this new composite system, we again trace the sequence of physical processes occurring subsequent to the excitation of the plate to define the inputs, outputs, and couplings associated with these subsystems. A schematic diagram of these processes is shown in figure 5-3.

A comparison of figures 5-2 and 5-3 reveals that the primary difference between the coupled subsystems that define the mathematical model for the displacement of the plate and those that define the mathematical model for the pressure field in the half space is the subsystem associated with the acoustic half space. As shown in figure 5-3, the output of the acoustic half-space subsystem is the pressure field over the entire half space. This output, in addition to being defined as the output of the composite plate-fluid system, is spatially filtered such that the pressure field at the surface of the plate

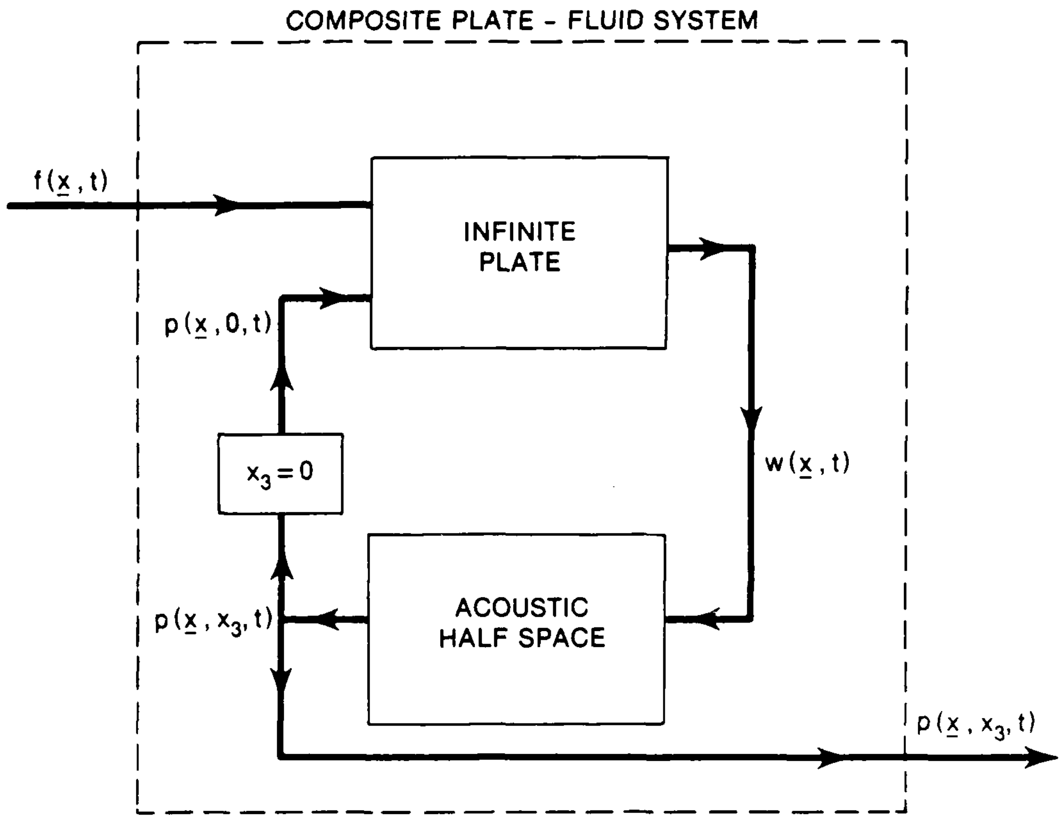

Figure 5-3. Schematic Systems Diagram for the Pressure Field Produced by the Forced Vibration of an Infinite Flat Plate

(i.e., at $x_3 = 0$) acts as an additional input to the plate. For the coupled system shown in figure 5-2, this spatial filtering was incorporated in the acoustic half-space subsystem so that the output of that subsystem was the pressure field at the surface of the plate, p(x̲,0,t).

By the above arguments, it is evident that the redefinition of the output of the composite plate-fluid system from the displacement field of the plate to the pressure field over the entire acoustic half space requires a redefinition of the acoustic half-space subsystem, but not of the infinite plate subsystem or the couplings between the subsystems. This redefined acoustic half-space subsystem is exactly that linear, time-invariant, space-limited,acoustic half-space system treated in section 4.3.3.1 of chapter 4. Inasmuch as one of the two coupled subsystems that forms the basis of the mathematical model for the pressure field produced by the forced

vibration of an infinite, flat plate is space varying, it follows that the redefined composite system is space varying.

The above examples demonstrate that changing the definition of the output from a specific assemblage of interacting physical components can effect a change in (1) the specification and classification of one or more of the single component subsystems used to mathematically model the multicomponent system and (2) the classification of the multicomponent system itself.

5.2 THE FREE RESPONSE OF COUPLED SYSTEMS

By definition, the free response of a multicomponent system is the self-sustained output of that system in the absence of any externally applied input. Recall that, in the absence of any input to the system, the initiation of the output field of the system cannot be addressed but, given certain knowledge of the output at any specific time, the output can be determined for all time.

Because the mathematical model of a multicomponent system is formulated by interpreting that composite system as an assemblage of coupled subsystems, the determination of the free response of the composite system is equivalent to determining the free response of the corresponding assemblage of coupled subsystems. The title of this section emphasizes this equivalence.

In this section, we demonstrate, by example, the procedure for formulating a mathematical model of, and obtaining a solution for, the free response of a multicomponent system. The multicomponent system addressed in this illustrative example is the displacement field of a free infinite plate subjected to fluid loading on one side. By comparing the free displacement field of the fluid-loaded plate to that of the plate in vacuo, we also examine the effects of the fluid loading on the free waves of the plate.

5.2.1. The Free Response of an Infinite Plate With Fluid Loading on One Side

The physical system and the corresponding assemblage of coupled subsystems associated with the forced vibration of a thin, infinite, flat plate with an

acoustic fluid on one side and a vacuum on the other were presented and discussed in section 5.1.3. The physical system of interest in this section is the response of that same plate, fluid loaded on one side, in the absence of the externally applied forcing field, $f(\underline{x},t)$. Thus, it follows that a mathematical model for the free displacement field of this fluid-loaded plate can be formulated from that assemblage of coupled subsystems illustrated in figure 5-2 with the forcing function, $f(\underline{x},t)$, set equal to zero.

By reference to figure 5-2, it is evident that, in the absence of an externally applied forcing field, the infinite plate subsystem represents the displacement field of the plate forced by the pressure field, $p(\underline{x},0,t)$, applied to the upper surface of the plate. The resulting displacement field of the plate is the output of the composite plate fluid system, as well as that of the infinite plate subsystem. The forced response of a damped, infinite plate is treated in section 3.4.6. However, recall from section 3.3 that the outputs of free systems having losses cannot be described in the wavevector frequency domain. To circumvent this difficulty, we assume the plate to be undamped. With this assumption and the recollection that the pressure field $p(\underline{x},0,t)$ is applied in the negative x_3 direction, it follows from equation (3-103) that the displacement field output from the infinite plate subsystem is governed by

$$D\nabla^4 w(\underline{x},t) + \mu \frac{\partial^2 w(\underline{x},t)}{\partial t^2} = -p(\underline{x},0,t) \qquad (5\text{-}1)$$

for all $\underline{x}$ and t. Recall that D denotes the flexural rigidity of the plate and μ represents the mass per unit area of the plate. In this text, our interest is confined to those displacement and pressure fields that can be expressed in the forms

$$w(\underline{x},t) = (2\pi)^{-3} \iint_{-\infty}^{\infty} W(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega \qquad (5\text{-}2)$$

and

$$p(\underline{x},0,t) = (2\pi)^{-3} \iint_{-\infty}^{\infty} P(\underline{k},0,\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega\ . \qquad (5\text{-}3)$$

By using equations (5-2) and (5-3) in equation (5-1) and by recalling that equation (5-1) applies for all $\underline{x}$ and t, it follows that the wavevector-frequency descriptions of the input and output fields of the infinite plate subsystem are related by

$$W(\underline{k},\omega) = \frac{-P(\underline{k},0,\omega)}{\{Dk^4 - \mu\omega^2\}} , \tag{5-4}$$

where $k = |\underline{k}| = \sqrt{k_1^2 + k_2^2}$. This relationship has the mathematical form shown, in equation (3-59), to be characteristic of space and time-invariant linear systems. Thus, we conclude that the infinite plate subsystem is space and time invariant.

As was stated in section 5.1.3, the subsystem associated with the acoustic fluid is a special case of the space-limited acoustic half space system treated in section 4.3.3.1. By reference to equations (4-141) and (4-145) of section 4.3.3.1, it is evident that the output, $p(\underline{x},0,t)$, of the acoustic half-space subsystem is governed by

$$\nabla^2 p(x,x_3,t) + \frac{\partial^2 p(\underline{x},x_3,t)}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2 p(\underline{x},x_3,t)}{\partial t^2} = 0 , \quad x_3 > 0 , \tag{5-5}$$

and

$$\frac{\partial p(\underline{x},0,t)}{\partial x_3} = -\rho \frac{\partial^2 w(\underline{x},t)}{\partial t^2} \tag{5-6}$$

for all x and t. Here, ∇^2 denotes the two-dimensional Laplacian operator, ρ and c are the respective density and speed of sound in the fluid, and $w(\underline{x},t)$ is the displacement of the fluid at the boundary $x_3 = 0$ in the direction normal to that boundary. Recall, however, from section 5.1.3, that the normal displacement of the fluid at $x_3 = 0$ is imposed by the displacement of the plate. Recall, further, that the pressure field in the half space must satisfy the causality condition that the pressure must either propagate away,

or decay with increasing distance, from the source of excitation of the pressure field: that is, the boundary common to the plate and fluid at $x_3 = 0$.

It is interesting to note, from equations (5-5) and (5-6), that to obtain a solution for the pressure field at $x_3 = 0$, one must first obtain a solution for the pressure throughout the entire half space inasmuch as the boundary condition (equation (5-6)) requires knowledge of the gradient of the pressure normal to the boundary.

The solution to equations (5-5) and (5-6), subject to the causality condition, were developed in section 4.3.3.1. By use of equation (4-159), it is straightforward to show that the wavevector-frequency description of the output field, $P(\underline{k},0,\omega)$, of the subsystem associated with the acoustic fluid is related to the wavevector-frequency description of the displacement field of the plate, $W(\underline{k},\omega)$, imposed at the boundary of the fluid by

$$P(\underline{k},0,\omega) = \begin{cases} \dfrac{i\rho\omega^2 W(\underline{k},\omega)}{k_0\sqrt{1 - k^2/k_0^2}} , & k \le |k_0| , \\[2ex] \dfrac{-\rho\omega^2 W(\underline{k},\omega)}{\sqrt{k^2 - k_0^2}} , & k > |k_0| , \end{cases} \tag{5-7}$$

where it will be recalled that $k_0 = \omega/c$. By reference to section 3.4.3, it can be verified that the input-output relationship described by equation (5-7) has a mathematical form consistent with that of a space- and time-invariant system.

The coupled set of equations (5-4) and (5-7) form the mathematical model, in the wavevector-frequency domain, for the displacement field of the free, infinite plate subjected to fluid loading on one side. By substituting equation (5-7) into equation (5-4), it is easily verified that the wavevector-frequency description of the displacement field of this fluid loaded plate is governed by

$$\left\{Dk^4 - \mu\omega^2 + \frac{i\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}}\right\}W(\underline{k},\omega) = 0 , \quad k \leq |k_0| , \tag{5-8}$$

and

$$\left\{Dk^4 - \mu\omega^2 - \frac{\rho\omega^2}{\sqrt{k^2 - k_0^2}}\right\}W(\underline{k},\omega) = 0 , \quad k > |k_0| . \tag{5-9}$$

By these equations, it is evident that unless

$$\left\{Dk^4 - \mu\omega^2 + \frac{i\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}}\right\} = 0 , \quad k \leq |k_0| , \tag{5-10}$$

$W(\underline{k},\omega)$ is zero over the wavevector range $k \leq |k_0|$, and unless

$$\left\{Dk^4 - \mu\omega^2 - \frac{\rho\omega^2}{\sqrt{k^2 - k_0^2}}\right\} = 0 , \quad k > |k_0| , \tag{5-11}$$

$W(\underline{k},\omega)$ is zero over the wavevector range $k > |k_0|$.

In equations (5-2) and (5-3), the components k_1 and k_2 of the wavevector $\underline{k}$ and the frequency ω are restricted to be real. Thus, we seek only those solutions to equations (5-10) and (5-11) for which both k and ω are real. With this restriction, it is evident that equation (5-10) is satisfied only if $k = \omega = 0$. This solution represents a static, rigid body displacement rather than a vibratory motion of the plate and can therefore be ignored. We thus conclude that equation (5-10) has no solutions of consequence to the dynamic motion of the plate in the real wavevector domain $k < |k_0|$, and therefore

$$W(\underline{k},\omega) = 0 , \; k \leq |k_0| . \tag{5-12}$$

Let us now seek the solutions of equation (5-11) for real values of ω and k. To accomplish this, we recall that the symbol $\sqrt{}$ denotes the positive

square root of a positive number, and we define the real, positive parameter

$$s = \sqrt{k^2 - k_0^2} \ , \quad k > |k_0| \ . \tag{5-13}$$

By using equation (5-13) and by recalling that $k_0 = \omega/c$, we can rewrite equation (5-11) in the form of the following cubic equation in s:

$$s^3 + \left(\frac{\rho}{\mu}\right)s^2 + \left(\frac{Dk^4}{\mu c^2} - 1\right)k^2 s - \left(\frac{\rho}{\mu}\right)k^2 = 0 \ . \tag{5-14}$$

Inasmuch as s is defined to be real and positive, we seek the real, positive roots of equation (5-14). It is well known that a cubic equation with real coefficients has either three real roots or one real root and two conjugate complex roots. It is also known[1] that if s_1, s_2, and s_3 denote the roots of equation (5-14), then

$$s_1 s_2 s_3 = \rho/\mu \tag{5-15}$$

and

$$s_1 + s_2 + s_3 = -\rho/\mu \ . \tag{5-16}$$

By equations (5-15) and (5-16), it is easily deduced that there is only one real, positive root of equation (5-14). If we denote that real, positive root by $s_1(k)$, it follows from equation (5-13) that the corresponding (and only) real roots of equation (5-11) are given by

$$\omega = \pm c\sqrt{k^2 - s_1^2(k)} = \pm\omega_1(k) \ . \tag{5-17}$$

Thus, it is evident that equation (5-11) has two real roots in ω, equal in magnitude and opposite in sign.

It is convenient to use equation (2-14) to define the phase speed, c_p', associated with $\omega_1(k)$ at any wavevector $\underset{\sim}{k}$ as

$$c_p'(k) = |\omega_1(k)|/|\underset{\sim}{k}| = |\omega_1(k)|/k \tag{5-18}$$

and then to rewrite equation (5-17) as

$$\omega = \pm k c_p'(k) \ . \tag{5-19}$$

The phase speed $c_p'(k)$ represents the speed of propagation of the free waves in the fluid-loaded infinite plate as a function of the magnitude of the wavevector characterizing that wave. By substitution of equation (5-19) into equation (5-11), it can be shown that the phase speed, $c_p'(k)$, is the positive, real root of

$$Dk^2 - \mu c_p'^2 - \frac{\rho c c_p'^2}{k\sqrt{c^2 - c_p'^2}} = 0 \ , \quad c^2 > c_p'^2 \ . \tag{5-20}$$

By the above arguments, we have established that equation (5-11) has only two real roots in ω. It can further be shown, by equations (5-13), (5-15), and (5-16), that equation (5-11) also has four other complex roots that occur in conjugate pairs. If we denote the product of the factors of equation (5-11) associated with these complex roots by $Q(k,\omega)$, then it follows, by use of equations (5-17) and (5-19), that equation (5-11) can be rewritten in the form

$$Dk^4 - \mu\omega^2 - \frac{\omega^2}{\sqrt{k^2 - k_0^2}} = (\omega^2 - k^2 c_p'^2)Q(k,\omega) = 0 \ , \quad k > |k_0| \ . \tag{5-21}$$

Consequently, equation (5-9) can be written

$$(\omega^2 - k^2 c_p'^2)Q(k,\omega)W(\underline{k},\omega) = 0 \ , \quad k > |k_0| \ . \tag{5-22}$$

To obtain a solution of equation (5-22) for $W(\underline{k},\omega)$, we argue as follows. By use of equation (2-50) and the arguments presented in section 3.3.1, it can be shown that

$$Q(k,\omega)W(k,\omega) = A(k)\delta(\omega - kc_p') + B(\underline{k})\delta(\omega + kc_p') \ , \quad k > |k_0| \ , \tag{5-23}$$

where $A(\underline{k})$ and $B(\underline{k})$ are unspecified functions of the wavevector k. Therefore, it follows that

$$W(\underline{k},\omega) = \frac{A(\underline{k})}{Q(k,\omega)}\,\delta(\omega - kc_p') + \frac{B(\underline{k})}{Q(k,\omega)}\,\delta(\omega + kc_p') \ , \quad k > |k_0| \ . \quad (5\text{-}24)$$

However, by making use of the sampling property of the Dirac delta function and recalling that c_p' is a function of k, we can write

$$\{A(\underline{k})/Q(k,\omega)\}\delta(\omega - kc_p') = \{A(\underline{k})/Q(k,kc_p')\}\delta(\omega - kc_p') = \alpha(\underline{k})\delta(\omega - kc_p') \quad (5\text{-}25)$$

and

$$\{B(\underline{k})/Q(k,\omega)\}\delta(\omega + kc_p') = \{B(\underline{k})/Q(k,-kc_p')\}\delta(\omega + kc_p') = \beta(\underline{k})\delta(\omega + kc_p') \ . \quad (5\text{-}26)$$

Consequently,

$$W(\underline{k},\omega) = \alpha(\underline{k})\delta(\omega - kc_p') + \beta(\underline{k})\delta(\omega + kc_p') \ , \quad k > |k_0| \ , \quad (5\text{-}27)$$

where $\alpha(\underline{k})$ and $\beta(\underline{k})$ are, as yet, unspecified functions of the wavevector $\underline{k}$.

By equations (5-12) and (5-27), we can conclude that the wavevector-frequency description of the displacement field of the fluid-loaded, infinite plate has the mathematical form

$$W(\underline{k},\omega) = \alpha(\underline{k})\delta(\omega - kc_p') + \beta(\underline{k})\delta(\omega + kc_p') \quad (5\text{-}28)$$

for all $\underline{k}$ and ω. The unspecified functions $\alpha(\underline{k})$ and $\beta(\underline{k})$ are determined by the initial conditions of the plate motion.

As we did in the case of the in-vacuo infinite plate, let us specify the initial displacement and velocity fields of the plate to be

$$w(\underline{x},0) = w_0(\underline{x}) \quad (5\text{-}29)$$

and

$$\frac{\partial w(\underline{x},0)}{\partial t} = v_0(\underline{x}) \ . \quad (5\text{-}30)$$

By use of equations (5-2), (5-28), (5-29), and (5-30), it is a simple matter to show that

$$W(\underline{k},\omega) = \pi W_0(\underline{k})\{\delta(\omega - kc_p') + \delta(\omega + kc_p')\}$$

$$+ \frac{\pi V_0(\underline{k})}{ikc_p'(k)} \{\delta(\omega - kc_p') - \delta(\omega + kc_p')\} \qquad (5\text{-}31)$$

for all $\underline{k}$ and ω, where $W_0(\underline{k})$ and $V_0(\underline{k})$ are the respective spatial Fourier transforms of $w_0(\underline{x})$ and $v_0(\underline{x})$.

It is instructive to compare the wavevector-frequency description of the displacement field of the freely vibrating infinite plate with fluid loading on one side, given by equation (5-31), with the corresponding description of the free vibration of the infinite plate in vacuo, given by equation (3-31). By use of equations (2-15) and (3-23), it is straightforward to show that the phase speed, c_p, of the in-vacuo plate can be expressed as

$$c_p(k) = k\sqrt{D/\mu} \ . \qquad (5\text{-}32)$$

Consequently, from equation (3-31), the wavevector-frequency description of the displacement field of the freely vibrating infinite plate in vacuo can be written in the form

$$W(\underline{k},\omega) = \pi W_0(\underline{k})\{\delta(\omega - kc_p) + \delta(\omega + kc_p)\} + \frac{\pi V_0(\underline{k})}{ikc_p(k)} \{\delta(\omega - kc_p) - \delta(\omega + kc_p)\} \ . \qquad (5\text{-}33)$$

Comparison of equations (5-31) and (5-33) reveals that the wavevector-frequency descriptions of the free vibration of the infinite, fluid-loaded plate and the infinite plate in vacuo differ only in the propagation speeds of the respective free waves. Therefore, the physical interpretation of equation (3-31) given in section 3.3.2 can be directly applied to equation (5-31) by properly accounting for this difference in phase speeds.

Inasmuch as the phase speed of a free wave in an infinite fluid-loaded plate differs from that in an infinite plate in vacuo, it follows that the

free wavenumber associated with the fluid-loaded plate must differ from that of the in-vacuo plate. Recall that the free wavenumber is defined as the magnitude of a wavevector associated with a free wave. If we denote a free wavenumber of the fluid-loaded plate by k_p', then it follows from equation (5-19) that

$$k_p'(\omega) = |\omega|/c_p' . \tag{5-34}$$

By equations (5-19), (5-20), and (5-34), the free wavenumber of the fluid-loaded plate is the real, positive root of

$$Dk_p'^4 - \mu\omega^2 - \frac{\rho\omega^2}{\sqrt{k_p'^2 - k_0^2}} = 0 , \quad k_p' > |k_0| . \tag{5-35}$$

In section 3.2.2, we showed the free wavenumber of the in-vacuo infinite plate, k_p, to be given by

$$k_p = \sqrt[4]{\mu\omega^2/D} . \tag{5-36}$$

In contrast to equation (5-36), equation (5-35) is of such complexity that a solution for k_p' can only be obtained by numerical techniques. However, by use of equation (5-36) and the mathematical form of equation (5-35), we can deduce that (1) k_p' must exceed k_p for all ω and (2) the fluid parameters appear in a term consistent, in form, with an additional mass, or inertial force. Thus, we conclude that the fluid loading acts as an additional mass to the plate, thereby (in accordance with equation (5-36)) increasing the free wavenumber of the plate.

To provide a quantitative example of the effect of fluid loading on the free wavenumber of a plate, figure 5-4 presents a comparison of the free wavenumber (k_p') of an infinite 2.54-cm-thick steel plate with water loading on one side with the free wavenumber (k_p) of the same plate in vacuo over the frequency range 0 to 12 kilohertz (kHz). Also included in this figure, for reference purposes, is the acoustic wavenumber (k_0) of the water.

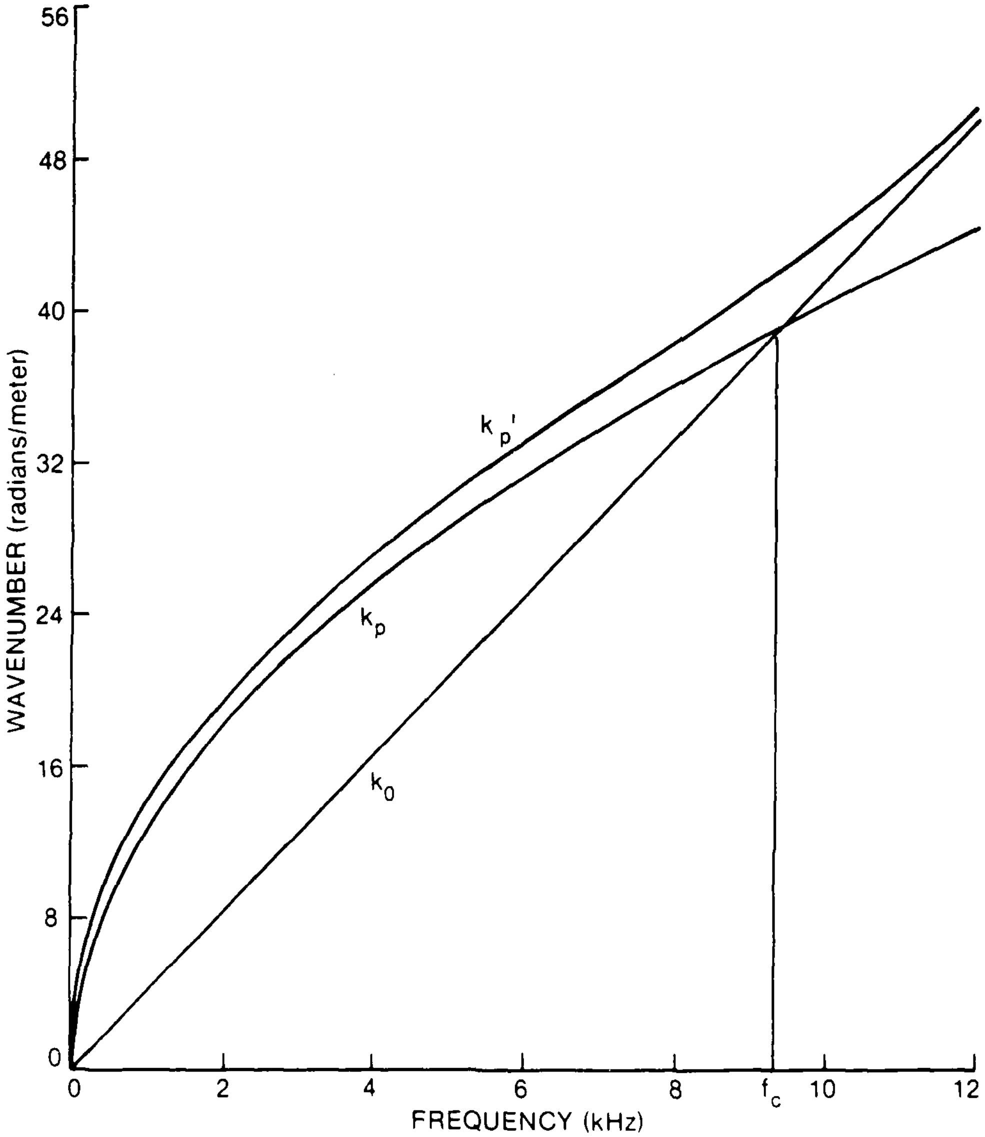

Figure 5-4. Comparison of In-Vacuo and Water-Loaded (One Side) Free Wavenumber of an Infinite 2.54-cm-Thick Steel Plate

Figure 5-4 shows that, in the frequency range where k_p is greater than k_0, the free wavenumber of the water-loaded plate is about 6 percent greater than that of the plate in vacuo. In the frequency range where k_p is less than k_0, the free wavenumber of the water-loaded plate asymptotically approaches k_0, and the ratio of k_p' to k_p increases with increasing frequency. Note that, as required by equation (5-35), k_p' exceeds k_0 at all frequencies.

Because the character of k_p' is similar to that of k_p at frequencies where $k_p > k_0$ and similar to k_0 at frequencies where $k_p < k_0$, the frequency at which $k_p = k_0$ is given the special name "coincidence frequency" and is denoted by f_c. By equation (5-36) and the definition of k_0, it is easily shown that the coincidence frequency is given by

$$f_c = \frac{c^2}{2\pi} \sqrt{\mu/D} \; . \tag{5-37}$$

For the example shown in figure 5-4, the coincidence frequency is about 9.4 kHz.

Junger and Feit[2] show that an "extremely accurate" approximation to k_p' can be obtained at frequencies below the coincidence frequency by replacing the k_p' under the radical in equation (5-35) by the free wavenumber of the in-vacuo infinite plate, k_p. With this substitution, we obtain the following approximation for the free wavenumber of the infinite plate, fluid loaded on one side:

$$k_p'(\omega) \approx k_p \sqrt[4]{1 + \frac{\rho}{\mu\sqrt{k_p^2 - k_0^2}}} \quad , \quad f < f_c \; . \tag{5-38}$$

Figure 5-5 compares the free wavenumber computed from equation (5-35) with the approximate value obtained by equation (5-38) for the same 2.54-cm-thick steel plate, water loaded on one side, that was characterized in figure 5-4. As is evident from figure 5-5, equation (5-38) provides an excellent approximation to the free wavenumber of the fluid-loaded plate over the frequency range $f \leq 0.8 f_c$.

By applying similar arguments to equation (5-20), it would seem reasonable that a good approximation to $c_p'(k)$ can be obtained, over the wavenumber range where $c_p(k) < c$, by replacing the c_p' under the square root by the phase speed of the in-vacuo plate, c_p. By making this substitution and using equation (5-32), we obtain

$$c_p'(k) \approx c_p(k) \Big/ \sqrt{1 + \frac{\rho c}{\mu k \sqrt{c^2 - c_p^2(k)}}} \quad , \quad k < k_c \tag{5-39}$$

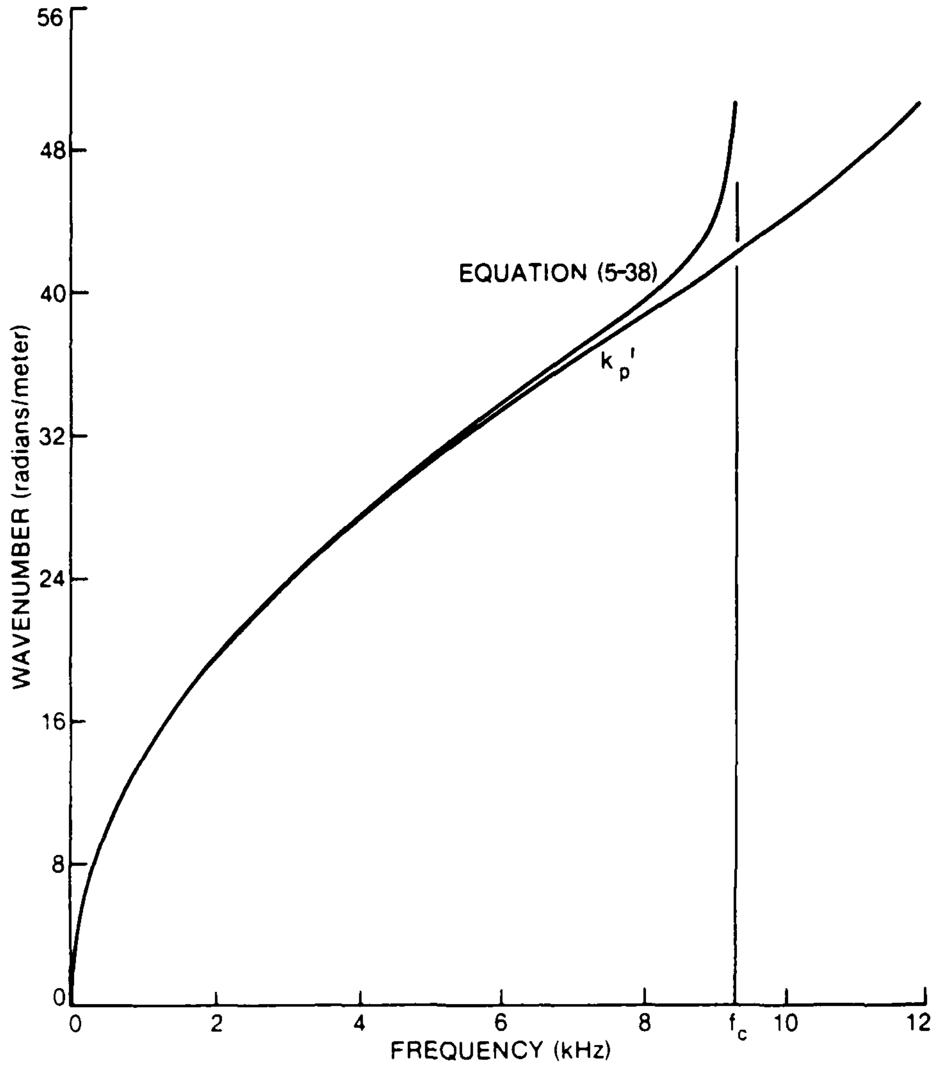

Figure 5-5. Comparison of Exact and Approximate Values of the Free Wavenumber of an Infinite 2.54-cm-Thick Steel Plate, Fluid Loaded on One side

Here, k_c denotes the critical wavenumber, defined as that wavenumber at which $c_p(k) = c$ and, by equation (5-32), given as

$$k_c = c\sqrt{\mu/D} \; . \tag{5-40}$$

Figure 5-6 compares the phase speed of the free wave computed from equation (5-20) with the approximate value computed from equation (5-39) for the 2.54-cm-thick infinite steel plate with water loading on one side. Here again, it is seen that equation (5-39) provides an excellent approximation

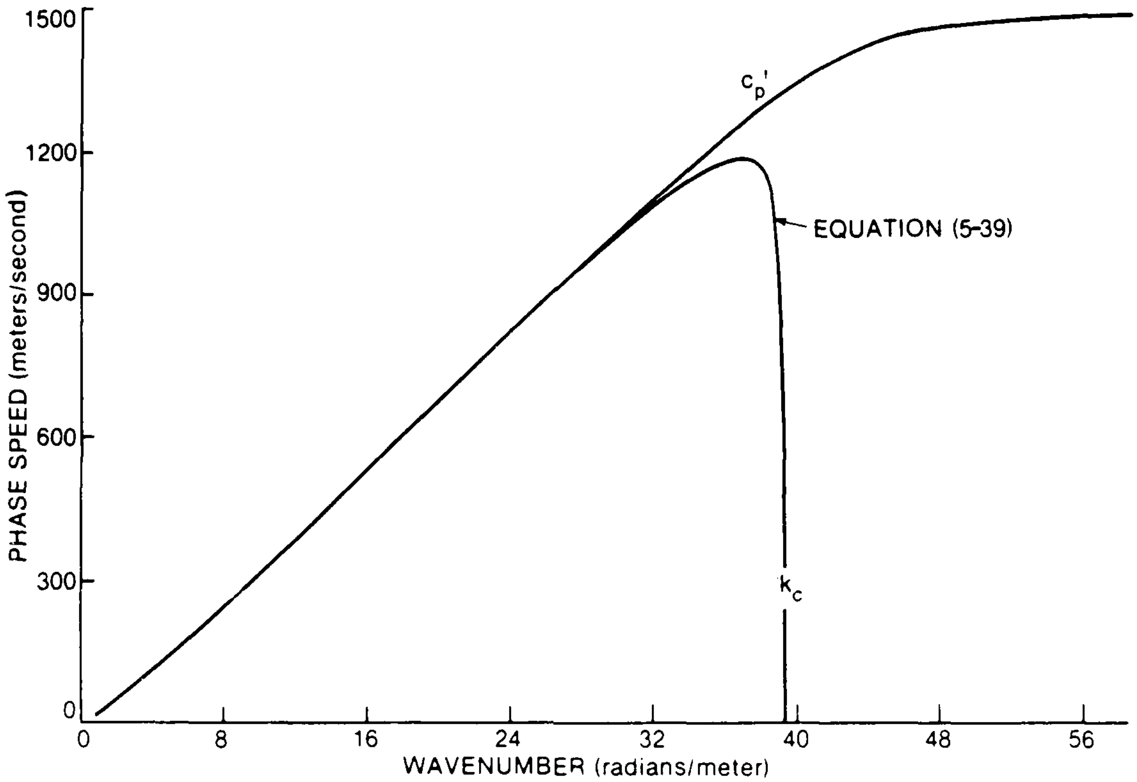

Figure 5-6. Comparison of Exact and Approximate Values of the Propagation Speed of the Free Wave in an Infinite 2.54-cm-Thick Steel Plate, Fluid Loaded on One Side

of the phase speed at wavenumbers less than 80 percent of the critical wavenumber.

As a final observation, it is easily shown, by equations (5-2) and (5-33), that the space-time description of the free vibration of the fluid-loaded infinite plate is given by

$$w(\underline{x},t) = (2\pi)^{-2} \iint_{-\infty}^{\infty} \left[\left\{ W_0(\underline{k}) - \frac{iV_0(\underline{k})}{kc_p'(k)} \right\} \exp\{i[\underline{k}\cdot\underline{x} + kc_p'(k)t]\} \right.$$

$$\left. + \left\{ W_0(\underline{k}) + \frac{iV_0(\underline{k})}{kc_p'(k)} \right\} \exp\{i[\underline{k}\cdot\underline{x} - kc_p'(k)t]\} \right] d\underline{k} \qquad (5\text{-}41)$$

for all x and t. However, owing to the dependence of c_p' on k, no further simplification of equation (5-41) is possible without specification of $W_0(\underline{k})$ and $V_0(\underline{k})$.

5.3 THE FORCED RESPONSE OF COUPLED SYSTEMS

Different multicomponent systems consist of different assemblages of physical components with different interactions between components and different inputs. By the arguments of section 5.1, such different multicomponent systems can be interpreted as different combinations of single component subsystems with different couplings between, and inputs to, the various subsystems. These coupled systems form the basis for the mathematical models of the corresponding multicomponent systems. Recall that although all systems (or subsystems) treated here are linear and time invariant, they can be either space varying or space invariant, as appropriate. By the arguments of chapters 3 and 4, it follows that the mathematical models of different multicomponent systems consist of different combinations of simultaneous, linear, homogeneous or inhomogeneous partial differential equations with different coupling conditions between equations. The coefficients of the various differential equations are time invariant, but may be either space invariant or space varying, as appropriate.

Clearly, by the above discussion, it is impractical to attempt to develop a general input-output relationship applicable to all multicomponent, or coupled, systems. Rather, the emphasis in this section will be to demonstrate, by example, techniques for formulating and solving mathematical models for the forced response of coupled systems. In the subsections to follow, we will address the forced response of two multicomponent systems: (1) the infinite plate subjected to fluid loading on one side and (2) the finite, simply supported plate subjected to fluid loading on one side. The forced response of these systems can then be compared with the forced response of the corresponding plates in vacuo to determine the effects of fluid loading on the forced response of infinite and finite plates.

5.3.1 The Forced Response of an Infinite Plate With Fluid Loading on One Side

This illustrative example is the forced version of the coupled system described and discussed in section 5.1.3. The geometry of the physical system is illustrated in figure 5-1, and the schematic diagram of the corresponding assemblage of coupled subsystems is illustrated in figure 5-2.

Let us first consider the subsystem associated with the infinite plate. By reference to figures 5-1 and 5-2, it is evident that the plate is subjected to two forcing fields: (1) the externally applied input, $f(\underline{x},t)$, and (2) the oppositely directed pressure field, $p(\underline{x},0,t)$, acting over the upper surface of the plate. The output of this subsystem, the displacement field, $w(\underline{x},t)$, of the plate, is also the output of the composite plate-fluid system. By equation (3-103), the response of an infinite plate (having flexural rigidity D, mass per unit area μ, and viscous damping per unit area r) to the forcing fields $f(\underline{x},t)$ and $-p(\underline{x},0,t)$ is governed by

$$D\nabla^4 w(\underline{x},t) + r\frac{\partial w(\underline{x},t)}{\partial t} + \mu\frac{\partial^2 w(\underline{x},t)}{\partial t^2} = f(\underline{x},t) - p(\underline{x},0,t) \qquad (5\text{-}42)$$

over all $\underline{x}$ and t.

Note that, for the treatment of the forced response of this composite system, we have assumed the plate to be damped, whereas in our treatment of the free response of the same composite system, the plate was assumed (for reasons explained in section 5.2.1) to be undamped. The reason for including damping in the model for the forced response of the plate subsystem is that its presence, as explained and illustrated in sections 3.4.4 and 3.4.5, simplifies causality arguments and permits unambiguous definition of the subsystem response over all wavevectors and frequencies.

By assuming that $w(\underline{x},t)$ and $p(\underline{x},0,t)$ exist in the forms of equations (5-2) and (5-3), respectively, and that $f(\underline{x},t)$ can be expressed as

$$f(\underline{x},t) = (2\pi)^{-3}\int_{-\infty}^{\infty} F(\underline{k},\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\, d\underline{k}\, d\omega \ , \qquad (5\text{-}43)$$

it is straightforward to show, from equation (5-42), that the wavevector-frequency descriptions of the input and output fields of the infinite plate subsystem are related by

$$W(\underline{k},\omega) = \frac{F(\underline{k},\omega) - P(\underline{k},0,\omega)}{Dk^4 - \mu\omega^2 + ir\omega} \ , \qquad (5\text{-}44)$$

where $k = \sqrt{k_1^2 + k_2^2}$. This input-output relation has the mathematical form characteristic of a space- and time-invariant linear system.

By inspection of figure 5-2, it is evident that, inasmuch as the input to the coupled plate-fluid system is applied to the plate, the subsystem associated with the acoustic fluid is the same for both the forced and free versions of the composite plate-fluid system. This acoustic half-space subsystem was shown, in section 5.2.1, to be governed by equations (5-5) and (5-6) in the space-time domain and by equation (5-7) in the wavevector-frequency domain.

By the coupled set of equations (5-7) and (5-44), it can be shown that the wavevector-frequency description of the forced response of the infinite plate, fluid loaded on one side, is given by

$$W(\underline{k},\omega) = \begin{cases} \dfrac{F(\underline{k},\omega)}{\left\{[Dk^4 - \mu\omega^2] + i\left[r\omega + \dfrac{\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}}\right]\right\}}, & k \le |k_0| , \\[2em] \dfrac{F(k,\omega)}{\left\{\left[Dk^4 - \mu\omega^2 - \dfrac{\rho\omega^2}{\sqrt{k^2 - k_0^2}}\right] + i[r\omega]\right\}}, & k > |k_0| . \end{cases} \tag{5-45}$$

Note that, over both ranges of wavevector magnitudes, this input-output relationship for the composite plate-fluid system has the algebraic form (see equation (3-59)) characteristic of space- and time-invariant systems. Thus, we conclude that the composite system of the infinite plate, fluid loaded on one side, is a space- and time-invariant linear system.

By recognizing the composite plate-fluid system to be space and time invariant, we can readily deduce, by equations (3-59) and (5-45), that the wavevector frequency response, $G(k,\omega)$, of the infinite plate, fluid loaded on one side, is given by

$$G(\underline{k},\omega) = \begin{cases} \dfrac{1}{\left\{[Dk^4 - \mu\omega^2] + i\left[r\omega + \dfrac{\rho\omega^2}{k_0\sqrt{1 - k^2/k_0^2}}\right]\right\}}, & k \leq |k_0| , \\[2ex] \dfrac{1}{\left\{\left[Dk^4 - \mu\omega^2 - \dfrac{\rho\omega^2}{\sqrt{k^2 - k_0^2}}\right] + i[r\omega]\right\}}, & k > |k_0| . \end{cases} \quad (5\text{-}46)$$

Recall that the wavevector-frequency response, $G(\underline{k},\omega)$, is the wavevector-frequency transform of the Green's function, $g(\underline{\xi},\tau)$. It is good practice, at this point, to ensure that $G(\underline{k},\omega)$ is the causal wavevector-frequency response: that is, the wavevector-frequency transform of the causal Green's function. In section 3.4.4, we established that $G(\underline{k},\omega)$ was a causal wavevector-frequency response if $\tilde{G}(\underline{k},\tau)$, the inverse Fourier transformation of $G(\underline{k},\omega)$ on ω, and its temporal derivatives were zero for $\tau < 0$. However, as is often the case in coupled systems, the form of equation (5-46) is of sufficient mathematical complexity that one is quickly discouraged from attempting the inverse Fourier transformation required to obtain $\tilde{G}(\underline{k},\tau)$. Consequently, we are motivated to address the question of the causality of the wavevector-frequency response of the fluid-loaded plate (given by equation (5-46)) by logical, rather than mathematical, arguments.

To this end, we submit the following arguments. The composite plate-fluid system can be interpreted, according to figure 5-2, as two subsystems arranged in a feedback loop. The input-output relationship for the plate subsystem (equation (5-44)) has the form shown (by equation (3-59)) to be characteristic of space- and time-invariant systems. From equations (3-59) and (5-44), it can easily be verified that the wavevector-frequency response of the plate subsystem is identical to that given by equation (3-114), which was shown to characterize the causal Green's function for the infinite plate in vacuo. The input-output relationship for the acoustic half-space subsystem was taken directly from equation (4-159) (with x_3 set to zero). The causality of this input-output relationship was addressed and ensured in its derivation. Inasmuch as (1) the input to the composite system is applied directly to the

plate subsystem, (2) the input-output relationships used to model both the plate and half-space subsystems in the wavevector-frequency domain are causal, (3) the causal output of each of the two subsystems is input directly to the other subsystem in a feedback loop, and (4) the output of the (causal) plate system is defined as the output of the composite plate-fluid system, it follows that the wavevector-frequency response of the composite plate-fluid system, described by equation (5-46), must be causal.

Let us now shift our focus to the response characteristics of the fluid-loaded plate in the wavevector-frequency domain. As explained in section 3.4.6, the wavevector-frequency response can be interpreted as the complex amplitude of the wave of the form $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$ output from a system as the result of excitation of the system by the unit amplitude wave, $\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}$. Note, by equation (5-46), that the wavevector-frequency response of the fluid-loaded plate, like that of the plate in vacuo (see section 3.4.6), depends only on the magnitude of the wavevector, and not on its direction. As was explained for the case of the infinite plate in vacuo, this independence of $G(\underline{k},\omega)$ on the direction of $\underline{k}$ is a reflection of the spatial invariance of the fluid-loaded infinite plate system. That is, for a unit amplitude harmonic wave excitation of the plate, the complex amplitude of the response of the fluid-loaded infinite plate depends only on the wavelength and frequency of excitation and is independent of the direction of propagation of the harmonic wave excitation.

The effects of the fluid loading on the wavevector-frequency response of an infinite plate can best be illustrated by comparing the response of a fluid-loaded plate with that of the same plate in vacuo. Figures 5-7(a) and (b) compare the magnitude and phase, respectively, of the wavevector-frequency response of an infinite plate, fluid loaded on one side, to that of an identical plate in vacuo as a function of the wavevector magnitude, k, at a fixed frequency, ω.

By figure 5-7(a), it is evident that the fluid loading has a significant effect on the magnitude of the wavevector-frequency response of the infinite plate. However, by use of equations (3-114) and (5-46), the reasons for the differences between the magnitudes of these wavevector-frequency responses can

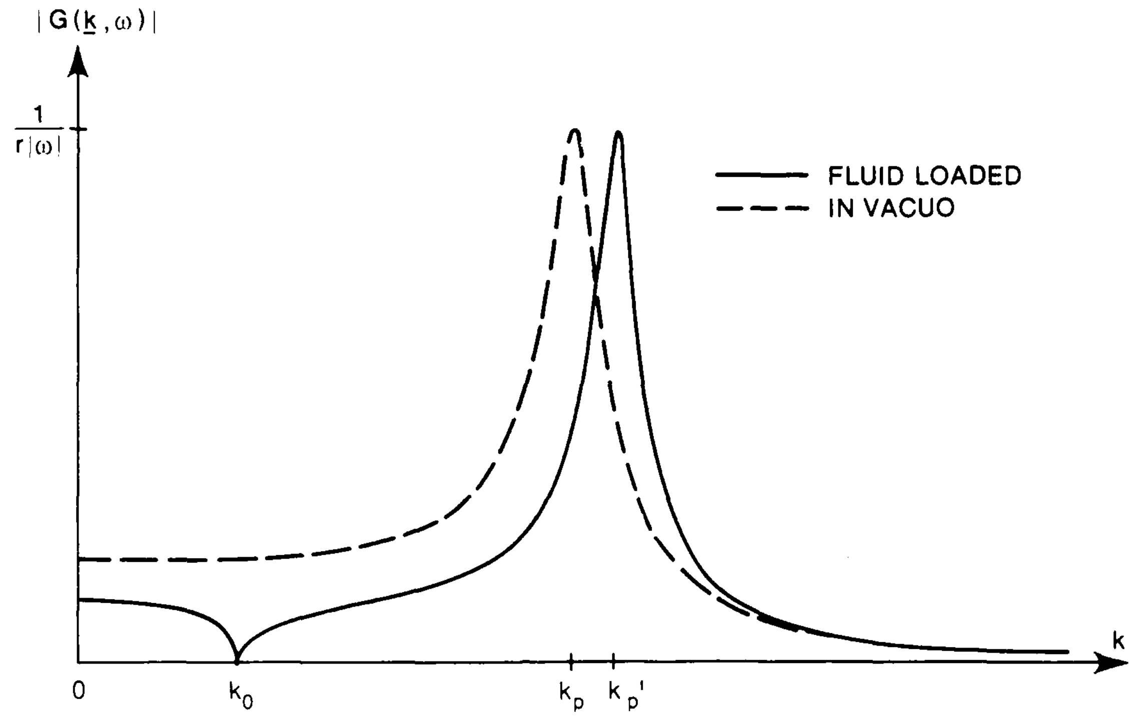

Figure 5-7(a). Comparison of the Magnitudes of the Wavevector-Frequency Responses of Fluid-Loaded and In-Vacuo Infinite Plates

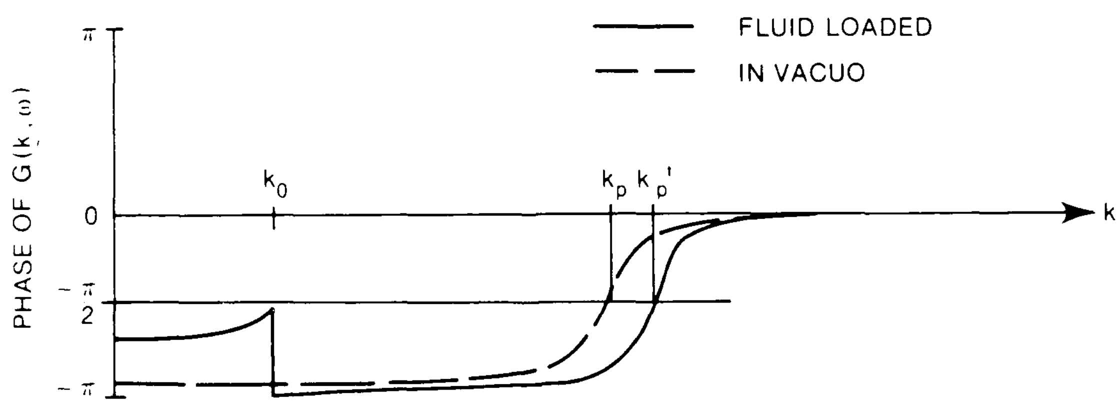

Figure 5-7(b). Comparison of the Phases of the Wavevector-Frequency Responses of Fluid-Loaded and In-Vacuo Infinite Plates

Figure 5-7. Comparison of the Wavevector-Frequency Responses of Fluid-Loaded and In-Vacuo Infinite Plates

easily be understood. Note first that, at wavenumber magnitudes less than the free wavenumber (k_p) of the in-vacuo plate, the magnitude of the wavevector-frequency response of the fluid-loaded plate is less than that of the plate in vacuo. This difference results from the additional damping (at wavevector magnitudes below k_0) or the additional mass (at wavevector magnitudes above k_0) imposed on the plate by the fluid in these respective wavenumber regimes. The amount of this difference in magnitude of response between fluid-loaded and in-vacuo plates in this wavenumber region can be shown, by equations (3-114) and (5-46), to increase as the quantity $\rho c/(\mu\omega)$, the ratio of the specific acoustic impedance of the fluid (ρc) to the inertial impedance of the plate ($\mu\omega$), increases. Thus, if the fluid impedance is small in comparison with the inertial impedance of the plate, the effect of the fluid loading on the magnitude of the wavevector-frequency response of the plate will be small (except at the wavenumber k_0). Conversely, if the specific acoustic impedance is large in comparison with the inertial impedance of the plate, the magnitude of the wavevector-frequency response of the fluid-loaded plate will be significantly lower than that of the plate in vacuo.

At the wavenumber k_0, the response of the fluid-loaded plate is seen to be zero, whereas that of the in-vacuo plate is nonzero. Recall, from section 4.3.3.1, that the impedance of the acoustic half space at the surface $x_3 = 0$ becomes infinite at wavevectors equal, in magnitude, to that of the acoustic wavenumber, $|k_0|$. Inasmuch as the plate motion and the fluid motion must be equal at the interface $x_3 = 0$, it follows that the impedance of the coupled plate-fluid system must also be infinite at wavevectors equal, in magnitude, to $|k_0|$. Consequently, the wavevector-frequency response of the fluid-loaded plate is zero at $k = k_0$. Conversely, equation (3-144) shows the wavevector-frequency response of the in-vacuo plate to be nonzero for all wavenumbers below the free wavenumber, k_p, of the plate.

Resonance in the wavevector-frequency responses of the in-vacuo and fluid-loaded plates occurs, at any fixed frequency, when the wavevector of excitation is equal, in magnitude, to the free wavenumbers, k_p and k_p', of the respective plates. As is evident in figure 5-7(a), the magnitude of the wavevector-frequency response at these resonance wavenumbers is equal to $1/(r|\omega|)$ for both the fluid-loaded and in-vacuo plates. However, because the

free wavenumber of the fluid-loaded plate is greater than that of the plate in vacuo, resonance occurs at a higher wavenumber in the fluid-loaded plate than in the in-vacuo plate.

For wavevector magnitudes large in comparison with the resonance wavenumbers, the wavevector-frequency responses of both the fluid-loaded and in-vacuo plates are governed by the flexural rigidity of the plate (i.e., the term Dk^4 in equations (3-114) and (5-46)). Inasmuch as figure 5-7(a) compares the magnitudes of the wavevector-frequency responses of identical plates, fluid loaded and in vacuo, it is not surprising that the magnitudes of the responses of the two plates are approximately equal at high wavenumbers.

Figure 5-7(b) compares the phases of the wavevector-frequency responses of the fluid-loaded and in-vacuo plates at the fixed frequency ω. As explained in section 3.4.6, the phase of the wavevector-frequency response can be interpreted as the phase of each harmonic wave component of the displacement field relative to that of the corresponding harmonic wave component of the excitation field. At wavevectors less, in magnitude, than the resonance wavenumber, k_p, the response of the in-vacuo plate is dictated primarily by inertial effects (i.e., the term $\mu\omega^2$ in equation (3-114)), and the displacement is nearly out of phase with the applied force. This same argument applies to the fluid-loaded plate for wavevector magnitudes greater than k_0 and less than k_p', where the inertia of the plate is augmented by the inertia associated with the fluid loading. For wavevectors less, in magnitude, than k_0, the fluid acts as additional damping to the plate, thereby reducing the phase lag between output and input relative to that shown for the in-vacuo plate. When the plate is excited by a harmonic wave characterized by a wavevector nearly equal (but less) in magnitude to that of the acoustic wavenumber, the damping force associated with the fluid loading becomes extremely large, and the displacement lags the applied force by 90 degrees. At resonance, the harmonic waves of displacement of both the fluid-loaded and in-vacuo plates lag the associated waves of applied force by 90 degrees. For harmonic wave excitations characterized by wavevectors larger, in magnitude, than the resonance wavenumber, the responses of both the fluid-loaded and in-vacuo plates are governed by the flexural rigidity of the plates (i.e., the term Dk^4 in the respective wavevector-frequency responses),

and the resulting wave of displacement is nearly in phase with the excitation.

The above example illustrates that the fluid-loaded plate responds most strongly to harmonic wave components of excitation that are characterized by wavevectors equal, in magnitude, to the free wavenumber of the fluid-loaded plate at the frequency of excitation. This behavior parallels that observed (in section 3.4.6) for the in-vacuo plate.

5.3.2 The Forced Response of a Finite, Simply Supported Plate With Fluid Loading on One Side

In this section, we develop the wavevector-frequency description of the forced displacement field of a finite, simply supported plate subjected to fluid loading on one side. The composite plate-fluid system of interest is illustrated in figure 5-8. Here, a thin plate (with flexural rigidity D, mass per unit area μ, damping coefficient per unit area r, and dimensions L_1 by L_2) is simply supported in a rigid baffle of infinite extent. The space above the plate and baffle, $x_3 > 0$, is occupied by an acoustic fluid having a density ρ and a speed of sound c. The space $x_3 < 0$ is vacuous. The plate is excited into motion by a force per unit area, $f(\underline{x},t)$, applied to the bottom surface of the plate. We wish to determine the displacement field of the plate resulting from this externally applied excitation.

A schematic diagram of this composite system is illustrated in figure 5-9. The baffled, simply supported plate is excited into motion by the externally applied forcing field, $f(\underline{x},t)$. The resulting displacement field, $w(\underline{x},t)$, is imposed on the fluid in the acoustic half space at the plate-fluid interface ($x_3 = 0$), thereby exciting a pressure field, $p(\underline{x},x_3,t)$, throughout the acoustic half space, $x_3 > 0$. This pressure field, acting over the top surface of the plate, produces an additional input field, $p(\underline{x},0,t)$, over the upper surface of the plate and baffle that acts opposite in direction to the externally applied forcing field. The output of this composite system is the displacement field of the plate, $w(\underline{x},t)$.

By figure 5-9, it is evident that the composite system of the fluid-loaded, simply supported plate can be interpreted as two coupled subsystems:

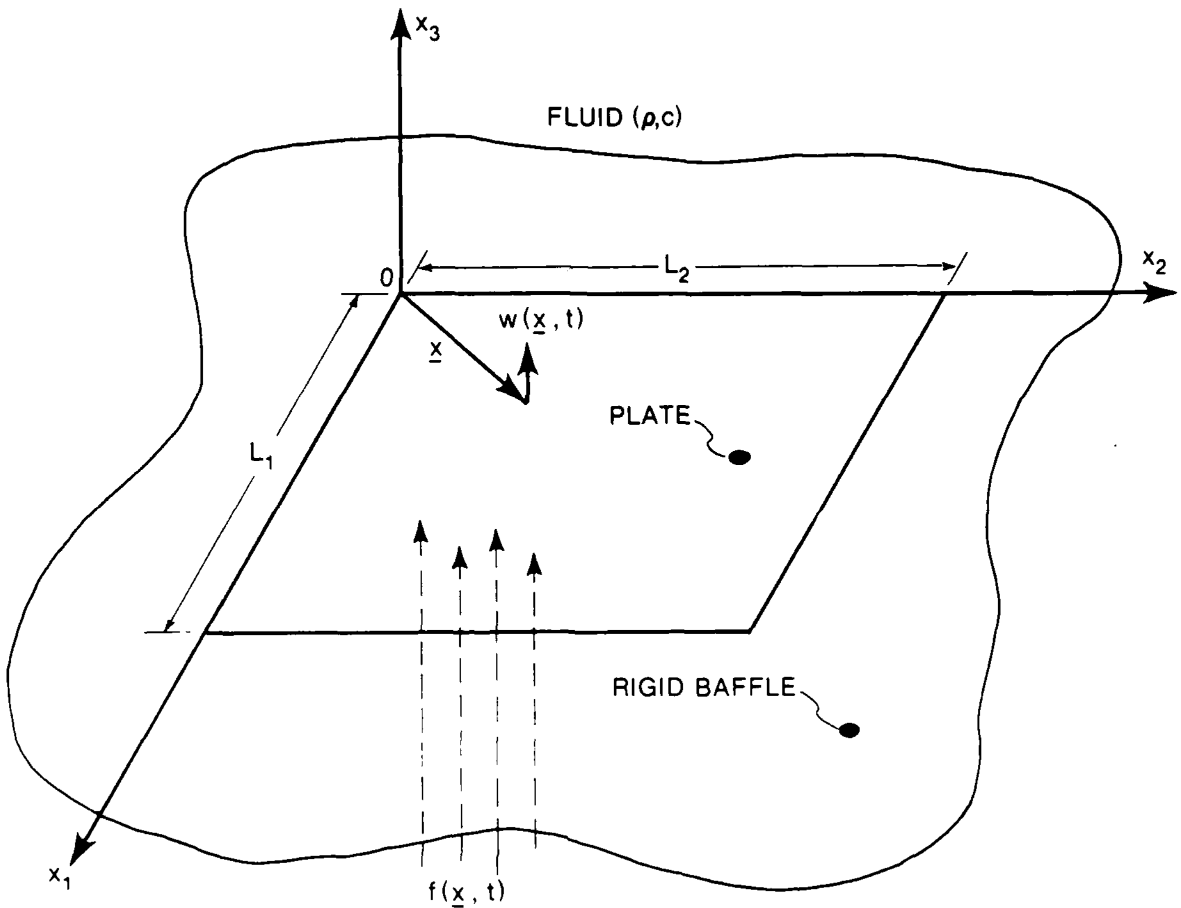

Figure 5-8. Geometry of the Finite, Simply Supported Plate With Fluid Loading on One Side

one subsystem represents the simply supported plate and the surrounding baffle and the second subsystem represents the acoustic half space. The couplings between these systems are identical to those occurring in the fluid-loaded infinite plate, which were described and discussed in section 5.1.3.

The reader might be justifiably curious as to why the simply supported plate and the surrounding rigid baffle are treated as a single subsystem. The answer is that, by including the rigid baffle in the subsystem associated with the simply supported plate, the input-output relationship for the plate-baffle subsystem can be directly obtained from that of the forced response of the simply supported plate, which was treated in section 4.3.3.2. That is, because the baffle is assumed to be rigid (i.e., of infinite impedance for all

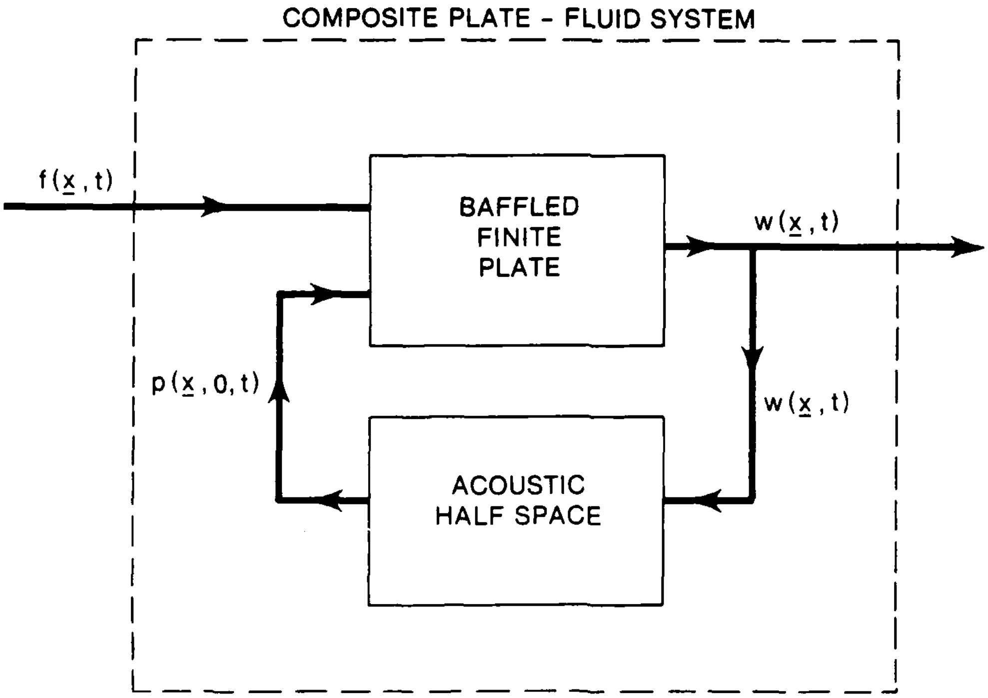

Figure 5-9. Schematic Diagram of Fluid-Loaded, Simply Supported Plate System

wavevectors and frequencies), the displacement, $w(\underline{x},t)$, normal to the plane of the plate and baffle is then known to be zero for all $\underline{x}$ outside the boundaries of the plate: that is, outside $0 < x_1 < L_1$ and $0 < x_2 < L_2$. This is exactly the displacement field that was assumed to exist outside the confines of the simply supported plate in the forced system treated in section 4.3.3.2. Thus, the motivation for including the rigid baffle in the subsystem associated with the simply supported plate was to enable us to use the relationships developed in section 4.3.3.2 to model this subsystem.

By reference to equation (4-210) of section 4.3.3.2, the displacement field output from the subsystem associated with the baffled, simply supported plate as a result of the forcing fields $f(\underline{x},t)$ and $-p(\underline{x},0,t)$ can be written as

$$w(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\underline{x},x_0,t - t_0)\{f(x_0,t_0) - p(x_0,0,t_0)\}\, dx_0\, dt_0 \qquad (5\text{-}47)$$

for all $\underline{x}$ and t. Here, $g(\underline{x},\underline{x}_0,t - t_0)$ is the exact and causal Green's function for the baffled, simply supported plate, which was shown, by equation (4-205), to be given by

$$g(\underline{x},\underline{x}_0,t - t_0) = \frac{2}{\pi L_1 L_2} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\beta(\underline{x})\alpha_{mn}(\underline{x})\beta(\underline{x}_0)\alpha_{mn}(\underline{x}_0)}{D[(m\pi/L_1)^2 + (n\pi/L_2)^2]^2 + ir\omega - \mu\omega^2} \right\}$$

$$\exp\{i\omega(t - t_0)\}\, d\omega \ , \tag{5-48}$$

where $\alpha_{mn}(\underline{x})$ are the normal modes of the simply supported plate defined by

$$\alpha_{mn}(\underline{x}) = \sin(m\pi x_1/L_1)\sin(n\pi x_2/L_2) \tag{5-49}$$

and $\beta(\underline{x})$ is the two-dimensional space-limiting function defined by

$$\beta(\underline{x}) = \{U(x_1) - U(x_1 - L_1)\}\{U(x_2) - U(x_2 - L_2)\} \ . \tag{5-50}$$

It should also be recalled (from section 4.3.3.2) that, in equation (5-47), $f(\underline{x},t)$ is equal to the force per unit area applied to the under surface of the plate in the spatial range $0 < x_1 < L_1$ and $0 < x_2 < L_2$, but can be arbitrarily specified outside this range inasmuch as forces outside the physical extent of the plate act only on the baffle, which is rigid.

It is straightforward to show, from equation (5-47), that the wavevector-frequency transform, $W(\underline{k},\omega)$, of the space-time displacement field is related to the wavevector-frequency descriptions of the forcing and surface pressure fields, $F(\underline{k},\omega)$ and $P(\underline{k},0,\omega)$, by

$$W(\underline{k},\omega) = (2\pi)^{-2} \int_{-\infty}^{\infty} G(\underline{k},-\underline{\sigma},\omega)\{F(\underline{\sigma},\omega) - P(\underline{\sigma},0,\omega)\}\, d\underline{\sigma} \ . \tag{5-51}$$

Here, $G(\underline{k},\underline{\sigma},\omega)$ is the two-wavevector-frequency response of the system (i.e., the multiple Fourier transform of $g(\underline{x},\underline{x}_0,\tau)$ on the variables $\underline{x}$, $\underline{x}_0$, and τ), which is given by

$$G(\underline{k},\underline{\sigma},\omega) = \frac{4}{L_1 L_2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_{mn}(\underline{k})I_{mn}(\underline{\sigma})}{Dk_{mn}^4 - \mu\omega^2 + ir\omega} \ , \tag{5-52}$$

where

$$k_{mn} = \sqrt{(m\pi/L_1)^2 + (n\pi/L_2)^2} \tag{5-53}$$

and

$$I_{mn}(\underline{k}) = \int_{-\infty}^{\infty} \beta(\underline{x})\alpha_{mn}(\underline{x})\exp(-i\underline{k}\cdot\underline{x})\, d\underline{x} \ . \tag{5-54}$$

Equation (5-51), with equation (5-52), defines the relationship between the wavevector-frequency descriptions of the input and the output fields for the subsystem associated with the baffled, simply supported plate. This input-output relationship has the mathematical form shown, in section 4.3.2, to be characteristic of a space-varying, time-invariant system.

The subsystem associated with the acoustic half space is identical to that used in the coupled system that represents the fluid-loaded infinite plate in sections 5.2.1 and 5.3.1. The causal relationship between the wavevector-frequency descriptions of the boundary displacement and the resulting surface pressure was given by equation (5-7). As was argued previously, this relationship has the mathematical form characteristic of a space- and time-invariant system.

The composite system of the fluid-loaded, simply supported plate surrounded by an infinite rigid baffle is mathematically modeled by the coupled set of equations (5-7) and (5-51). By substitution of equation (5-7) into equation (5-51), it can be shown that the wavevector-frequency description of the forced displacement of the baffled, simply supported plate, fluid loaded on one side, is governed by

$$W(\underline{k},\omega) = (2\pi)^{-2} \left\{ \int_{-\infty}^{\infty} G(\underline{k},-\underline{\sigma},\omega)\ F(\underline{\sigma},\omega)\ d\underline{\sigma} - i\rho c\omega \int_{|\underline{\sigma}| \le |k_0|} \frac{G(\underline{k},-\underline{\sigma},\omega)W(\underline{\sigma},\omega)}{\sqrt{1 - \sigma^2/k_0^2}}\, d\underline{\sigma} \right.$$

$$\left. + \rho\omega^2 \int_{|\underline{\sigma}| > |k_0|} \frac{G(\underline{k},-\underline{\sigma},\omega)W(\sigma,\omega)}{\sqrt{\sigma^2 - k_0^2}}\, d\underline{\sigma} \right\} , \tag{5-55}$$

where $\sigma^2 = \sqrt{\sigma_1{}^2 + \sigma_2{}^2}$. Note, by equation (5-55), that the input-output relationship for the fluid-loaded, baffled, simply supported plate in the wavevector-frequency domain is expressed in the form of an integral equation for $W(\underline{k},\omega)$. This mathematical form of input-output relationship was shown, in section 4.3.1.2, to be characteristic of space-limited, time-invariant linear systems.

The solution of the integral equation (5-55) for $W(\underline{k},\omega)$, subject to the constraints imposed by the simply supported boundary conditions and the surrounding rigid baffle, presents a formidable mathematical challenge. However, a conceptually simple (although computationally inefficient) approach to obtaining a solution for $W(\underline{k},\omega)$ is to assume that the space-time displacement and forcing fields can be expressed as a weighted superposition of the in-vacuo normal modes of the simply supported plate. That is, assume that the $w(\underline{x},t)$ and $f(\underline{x},t)$ can be expressed as

$$w(\underline{x},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overset{\nabla}{W}_{mn}(\omega)\beta(\underline{x})\alpha_{mn}(\underline{x})\exp(i\omega t)\, d\omega \tag{5-56}$$

and

$$f(\underline{x},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overset{\nabla}{F}_{mn}(\omega)\beta(\underline{x})\alpha_{mn}(\underline{x})\exp(i\omega t)\, d\omega\ . \tag{5-57}$$

As expressed in the form of equation (5-56), the displacement field of the baffled, fluid-loaded plate satisfies the simply supported boundary conditions and the requirement that the displacement be zero over the surface of the rigid baffle. The form of equation (5-57) reflects the fact that the externally applied forcing field acts only over the surface of the plate.

By assuming that the space-time displacement and forcing fields exist in the forms of equations (5-56) and (5-57), it follows (by use of equation (5-54)) that $W(\underline{k},\omega)$ and $F(\underline{k},\omega)$ can be written in the form

$$W(\underline{k},\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overset{\nabla}{W}_{mn}(\omega) I_{mn}(\underline{k}) \tag{5-58}$$

and

$$F(\underline{k},\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overset{\nabla}{F}_{mn}(\omega) I_{mn}(\underline{k}) \ . \tag{5-59}$$

By substituting equations (5-52), (5-58), and (5-59) into equation (5-55), we obtain the following relationship between the frequency-dependent modal coefficients of the displacement and forcing fields:

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overset{\nabla}{W}_{mn}(\omega) I_{mn}(\underline{k}) \\
&= \frac{1}{\pi^2 L_1 L_2} \left\{ \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{u=1}^{\infty} \sum_{v=1}^{\infty} \frac{\overset{\nabla}{F}_{pq}(\omega) I_{uv}(\underline{k})}{Dk_{uv}^{4} - \mu\omega^2 + ir\omega} \int_{-\infty}^{\infty} I_{pq}(\underline{\sigma}) I^*_{uv}(\underline{\sigma}) \, d\underline{\sigma} \right. \\
&\quad - i\rho c\omega \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{u=1}^{\infty} \sum_{v=1}^{\infty} \frac{\overset{\nabla}{W}_{mn}(\omega) I_{uv}(\underline{k})}{Dk_{uv}^{4} - \mu\omega^2 + ir\omega} \int_{|\underline{\sigma}| \le |k_0|} \frac{I_{mn}(\underline{\sigma}) I^*_{uv}(\underline{\sigma})}{\sqrt{1 - \sigma^2/k_0^2}} \, d\underline{\sigma} \\
&\quad \left. + \rho\omega^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{u=1}^{\infty} \sum_{v=1}^{\infty} \frac{\overset{\nabla}{W}_{mn}(\omega) I_{uv}(\underline{k})}{Dk_{uv}^{4} - \mu\omega^2 + ir\omega} \int_{|\underline{\sigma}| > |k_0|} \frac{I_{mn}(\underline{\sigma}) I^*_{uv}(\underline{\sigma})}{\sqrt{\sigma^2 - k_0^2}} \, d\underline{\sigma} \right\} .
\end{aligned} \tag{5-60}$$

By multiplying equation (5-60) by $I^*_{ij}(\underline{k})$ and integrating over all $\underline{k}$, we can employ the orthogonality condition (derived in section 4.2.2)

$$\int_{-\infty}^{\infty} I_{mn}(\underline{k}) I^*_{qs}(\underline{k}) \, d\underline{k} = \pi^2 L_1 L_2 \delta_{mq} \delta_{ns} \tag{5-61}$$

to show that

$$
\overset{\nabla}{W}_{ij}(\omega) = \frac{\overset{\nabla}{F}_{ij}(\omega)}{Dk_{ij}^{4} - \mu\omega^{2} + ir\omega}
$$

$$
- i \frac{\rho c \omega}{\pi^{2} L_{1} L_{2}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\overset{\nabla}{W}_{mn}(\omega)}{Dk_{ij}^{4} - \mu\omega^{2} + ir\omega} \int_{|\underline{\sigma}| \leq |k_{0}|} \frac{I_{mn}(\underline{\sigma}) I^{*}_{ij}(\underline{\sigma})}{\sqrt{1 - \sigma^{2}/k_{0}^{2}}} \, d\underline{\sigma}
$$

$$
+ \frac{\rho\omega^{2}}{\pi^{2} L_{1} L_{2}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\overset{\nabla}{W}_{mn}(\omega)}{Dk_{ij}^{4} - \mu\omega^{2} + ir\omega} \int_{|\underline{\sigma}| > |k_{0}|} \frac{I_{mn}(\underline{\sigma}) I^{*}_{ij}(\underline{\sigma})}{\sqrt{\sigma^{2} - k_{0}^{2}}} \, d\underline{\sigma} \; . \tag{5-62}
$$

If we then define

$$
r_{mnqs}(\omega) = \frac{\rho c}{\pi^{2} L_{1} L_{2}} \int_{|\underline{\sigma}| \leq |k_{0}|} \frac{I^{*}_{mn}(\underline{\sigma}) I_{qs}(\underline{\sigma})}{\sqrt{1 - \sigma^{2}/k_{0}^{2}}} \, d\underline{\sigma} \tag{5-63}
$$

and

$$
\mu_{mnqs}(\omega) = \frac{\rho}{\pi^{2} L_{1} L_{2}} \int_{|\underline{\sigma}| > |k_{0}|} \frac{I^{*}_{mn}(\underline{\sigma}) I_{qs}(\underline{\sigma})}{\sqrt{\sigma^{2} - k_{0}^{2}}} \, d\underline{\sigma} \; , \tag{5-64}
$$

equation (5-62) can be written

$$
\left\{ Dk_{mn}^{4} - \mu\omega^{2} + ir\omega \right\} \overset{\nabla}{W}_{mn}(\omega) + \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} \left[-\omega^{2} \mu_{mnqs}(\omega) + i\omega r_{mnqs}(\omega) \right] \overset{\nabla}{W}_{qs}(\omega) = \overset{\nabla}{F}_{mn}(\omega) \; . \tag{5-65}
$$

By recalling that the free wavenumber of the space- and time-invariant plate in vacuo is defined by

$$
k_{p}(\omega) = \sqrt[4]{\mu\omega^{2}/D} \; , \tag{5-66}
$$

we can rewrite equation (5-65) in the form

$$\sum_{q=1}^{\infty} \sum_{s=1}^{\infty} \left\{\delta_{mq}\delta_{ns}k_{qs}^{4} - k_p^{4}\left[\delta_{mq}\delta_{ns} + \frac{\mu_{mnqs}(\omega)}{\mu}\right]\right.$$

$$\left. + i\,\frac{r\omega}{D}\left[\delta_{mq}\delta_{ns} + \frac{r_{mnqs}(\omega)}{r}\right]\right\}\overset{\nabla}{W}_{qs}(\omega) = \frac{\overset{\nabla}{F}_{mn}(\omega)}{D} \, . \qquad (5\text{-}67)$$

Equation (5-67) represents a doubly infinite set of coupled equations for the frequency-dependent modal coefficients of the plate displacement, $\overset{\nabla}{W}_{mn}(\omega)$, in terms of the frequency-dependent modal coefficients of the external forcing field, $\overset{\nabla}{F}_{mn}(\omega)$. Owing to the coupled nature of these equations, a single modal component of the forcing field excites many modal components of displacement. This behavior is in sharp contrast to that observed in the forced response of the simply supported plate in vacuo (see equation (4-197) of section 4.3.3.2), where each modal component of the forcing field excited only the corresponding modal component of displacement. Clearly then, the coupling between a single modal component of force and the infinite set of modal coefficients of the displacement results from the fluid loading of the plate.

The fluid loading of the plate is applied by the pressure field (that is induced throughout the acoustic half space by the motion of the plate) acting on the upper surface of the plate. By equation (4-146) of section 4.3.3.1, the space-time description of the pressure field over the surface $x_3 = 0$ can be expressed as

$$p(\underline{x},0,t) = (2\pi)^{-3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\underline{k},0,\omega)\exp\{i(\underline{k}\cdot\underline{x} + \omega t)\}\,d\underline{k}\,d\omega \qquad (5\text{-}68)$$

for all $\underline{x}$ and t. However, over the surface of the plate, $0 \le x_1 \le L_1$ and $0 < x_2 < L_2$, $p(\underline{x},0,t)$ can be expressed in the form

$$p(x,0,t) \quad (2\pi)^{-1} \int_{-\infty}^{\infty} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \overset{\nabla}{P}_{mn}(\omega)\alpha_{mn}(x)\exp(i\omega t)\,d\omega \, , \quad \begin{array}{l} 0 \le x_1 \le L_1 \, , \\ 0 \le x_2 \le L_2 \, . \end{array} \qquad (5\text{-}69)$$

By use of equations (5-7), (5-58), (5-68), and (5-69), it follows that

$$\sum_{u=1}^{\infty}\sum_{v=1}^{\infty} \overset{\nabla}{P}_{uv}(\omega)\alpha_{uv}(\underline{x})$$

$$= (2\pi)^{-2}\sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\left\{\int_{|\underline{k}|\le|k_0|} \frac{i\rho c\omega\, \overset{\nabla}{W}_{qs}(\omega) I_{qs}(\underline{k})}{\sqrt{1 - k^2/k_0^{\,2}}} \exp[i(\underline{k}\cdot\underline{x})]\, d\underline{k} - \int_{|\underline{k}|>|k_0|} \frac{\rho\omega^2 \overset{\nabla}{W}_{qs}(\omega) I_{qs}(\underline{k})}{\sqrt{k^2 - k_0^{\,2}}} \exp[i(\underline{k}\cdot\underline{x})]\, d\underline{k}\right\} \tag{5-70}$$

for $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$. By multiplying equation (5-70) by $\alpha_{mn}(x)$ and integrating over $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$, we can use equations (4-54), (5-54), (5-63), and (5-64) to show that

$$\overset{\nabla}{P}_{mn}(\omega) = \sum_{q=1}^{\infty}\sum_{s=1}^{\infty}\{i\omega r_{mnqs}(\omega) - \omega^2\mu_{mnqs}(\omega)\}\overset{\nabla}{W}_{qs}(\omega) \; . \tag{5-71}$$

Equation (5-71), which defines the complex, frequency-dependent amplitude of the mn-th modal component of the pressure field acting over the upper surface of the plate, clearly shows that a single modal component of the displacement field, $\overset{\nabla}{W}_{qs}(\omega)$, produces an infinite number of modal components of pressure on the upper surface of the plate. These modal components of pressure, in turn, excite the corresponding modes of displacement of the plate. This phenomenon is known as modal coupling.

The presence and degree of modal coupling is dictated, in both equations (5-67) and (5-71), by the frequency-dependent quantities r_{mnqs} and μ_{mnqs}, defined by equations (5-63) and (5-64), respectively. It is desirable to interpret the roles of these quantities on the physics of the plate motion.

It is easily shown, by use of equations (5-54), (5-63), and (5-64), that r_{mnqs} and μ_{mnqs} are real quantities. Further, by equation (5-71), it is

evident that the combination of terms

$$\{i\omega r_{mnqs}(\omega) - \omega^2 \mu_{mnqs}(\omega)\} \overset{\nabla}{W}_{qs}(\omega)$$

can be interpreted, at any given frequency, as the contribution to the mn-th modal component of the pressure acting over the upper surface of the plate resulting from the qs-th modal component of the displacement field of the plate. More specifically, it can be shown by use of equation (5-56) that $i\omega\overset{\nabla}{W}_{qs}(\omega)$ can be interpreted as the complex frequency-dependent amplitude of the qs-th modal component of the normal velocity field of the plate. Therefore, because r_{mnqs} is real, the term $i\omega r_{mnqs}(\omega)\overset{\nabla}{W}_{qs}(\omega)$ describes a contribution to the mn-th modal component of pressure that is proportional to the qs-th modal component of the velocity field of the plate. Inasmuch as forces per unit area that are proportional to velocity are normally associated with losses (e.g., damping) to the system, the term r_{mnqs} can be interpreted as additional damping per unit area of the plate resulting from the coupling between mn-th and qs-th in-vacuo modes of the plate caused by the presence of the fluid. This interpretation is supported by equation (5-63), which shows that $r_{mnqs}(\omega)$ is proportional to an integral over the supersonic (i.e., $|\underline{\sigma}| \leq |k_0|$) wavevector components of the acoustic half space. Recall, from section 4.3.3.1, that these supersonic wavevector components are associated with waves that propagate away from the boundary (the plate-baffle surface) and, thereby, represent a loss mechanism to the plate.

By similar reasoning, the term $-\omega^2\overset{\nabla}{W}_{qs}(\omega)$ can be interpreted as the complex amplitude of the qs-th modal component of the acceleration field of the plate in a direction normal to its surface. Because μ_{mnqs} is real, the term $-\omega^2\mu_{mnqs}(\omega)\overset{\nabla}{W}_{qs}(\omega)$ represents a contribution to the mn-th modal component of the pressure acting over the upper surface of the plate that is proportional to the qs-th modal component of the acceleration field of the plate. Because forces per unit area that are proportional to acceleration are associated with the inertia (or mass) of the system, the term μ_{mnqs} can be interpreted as an additional mass imposed on the plate as a result of the coupling of the mn-th and qs-th modes caused by the presence of the fluid. As seen by equation (5-64), this additional mass is proportional to an integral of the subsonic (i.e., $|\underline{\sigma}| > |k_0|$) wavevector components of the acoustic

half space. These subsonic components were shown, in section 4.3.3.1, to be associated with waves that did not propagate away from the boundary, but decayed in amplitude with increasing distance from the boundary, (i.e., the plate). As these waves do not propagate away from the plate, they do not represent a loss mechanism to the plate. Rather they can only represent reactive forces (i.e., inertia or stiffness) on the plate. Davies[3] argued that the terms containing μ_{mnqs} lead to additional virtual mass of the plate. Therefore, μ_{mnqs} is interpreted as additional mass to the plate rather than additional stiffness.

By the above arguments, we conclude that the modal coupling, introduced by the physical coupling between the plate and the fluid, has the effect of adding mass and damping to the simply supported plate. The reader will recall that the effects of fluid loading on the space-invariant, infinite plate were to increase the apparent mass and damping of the plate. Thus, we see that the effects of fluid loading are similar between finite, simply supported plates and infinite, space-invariant plates.

If we define

$$A_{mnqs}(\omega) = \delta_{mq}\delta_{ns}k_{qs}^{4} - k_p^{4}\left[\delta_{mq}\delta_{ns} + \frac{\mu_{mnqs}(\omega)}{\mu}\right]$$

$$+ i\,\frac{r\omega}{D}\left[\delta_{mq}\delta_{ns} + \frac{r_{mnqs}(\omega)}{r}\right] , \qquad (5\text{-}72)$$

then equation (5-67) can be rewritten as

$$\sum_{q=1}^{\infty}\sum_{s=1}^{\infty} A_{mnqs}(\omega)\overset{\nabla}{W}_{qs}(\omega) = \frac{\overset{\nabla}{F}_{mn}(\omega)}{D} . \qquad (5\text{-}73)$$

In principle, although not necessarily in practice, a four-dimensional, frequency-dependent matrix, $B_{mnqs}(\omega)$, can be found that satisfies the relationship

$$\sum_{q=1}^{\infty}\sum_{s=1}^{\infty} B_{ijmn}(\omega)A_{mnqs}(\omega) = \delta_{iq}\delta_{js} . \qquad (5\text{-}74)$$

That is, the matrix $B_{mnqs}(\omega)$ is the inverse of the matrix $A_{mnqs}(\omega)$ at the frequency ω. By use of equations (5-73) and (5-74), it follows that $\overset{\nabla}{W}_{mn}(\omega)$ has the form

$$\overset{\nabla}{W}_{mn}(\omega) = \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} B_{mnqs}(\omega) \frac{\overset{\nabla}{F}_{qs}(\omega)}{D} . \tag{5-75}$$

Therefore, by equation (5-58), the wavevector-frequency description of the displacement field of the baffled, simply supported plate, fluid loaded on one side, has the form

$$W(\underline{k},\omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} B_{mnqs}(\omega) \frac{\overset{\nabla}{F}_{qs}(\omega)}{D} I_{mn}(\underline{k}) . \tag{5-76}$$

Equation (5-76) shows that, at any fixed frequency ω, the wavevector dependence of the displacement field of the baffled, fluid-loaded, simply supported plate is specified by a weighted summation of the wavevector transforms of the space-limited natural modes of the in-vacuo plate, $I_{mn}(\underline{k})$, over all mode numbers, m and n. By reference to equation (4-217) of section 4.3.3.2, it is evident that the wavevector dependence of the displacement field of the in-vacuo, simply supported plate is also specified by a weighted summation of $I_{mn}(\underline{k})$ over all m and n. However, a comparison of equations (4-217) and (5-76) shows that the modal weights applied to $I_{mn}(\underline{k})$ are considerably more complicated for the fluid-loaded plate than for the in-vacuo plate. Determination of the modal weights, $\overset{\nabla}{W}_{mn}(\omega)$, for the fluid-loaded, simply supported plate requires, by equation (5-75), knowledge of $B_{mnqs}(\omega)$, the inverse of $A_{mnqs}(\omega)$. Inasmuch as the mode numbers, m and n, range from 1 to infinity, it is evident that $B_{mnqs}(\omega)$ cannot be determined exactly. Therefore, only approximate solutions can be obtained for the wavevector-frequency (or space-time, for that matter) characteristics of the baffled, simply supported, fluid-loaded plate. A variety of such approximate solutions are presented by Junger and Feit[4] and Davies.[5] The majority of these approximate solutions require arguments too complex and lengthy to be presented here. However, to provide some insight into the response characteristics of the baffled, fluid-loaded, simply supported plate, we will

present one, somewhat simplistic, example of an approximate solution: the example of a light fluid loading.

Light fluid loading is defined as that situation in which the modal amplitude of the pressure, $\overset{\nabla}{P}_{mn}(\omega)$, defined by equation (5-71), is small in comparison with the externally applied modal force, $\overset{\nabla}{F}_{mn}(\omega)$. Thus, in the case of light fluid loading, the forces resulting from modal coupling are small, and each modal component of the applied force is primarily balanced by the modal forces associated with the stiffness and inertia of the plate. Under these assumptions, equation (5-65) can be rewritten

$$\left\{Dk_{mn}^{4} - [\mu + \mu_{mnmn}(\omega)]\omega^{2} + i[r + r_{mnmn}(\omega)]\omega\right\}\overset{\nabla}{W}_{mn}(\omega) + \varepsilon_{mn}(\omega) = \overset{\nabla}{F}_{mn}(\omega) \;, \tag{5-77}$$

where $\varepsilon_{mn}(\omega)$ denotes the contribution to the modal pressure resulting from the sum of the crosscoupled terms (i.e., $q \neq m$ and $s \neq n$) in equation (5-71). That is,

$$\varepsilon_{mn}(\omega) = \sum_{\substack{q=1 \\ q\neq m}}^{\infty} \sum_{\substack{s=1 \\ s\neq n}}^{\infty} [-\omega^{2}\mu_{mnqs}(\omega) + i\omega r_{mnqs}(\omega)]\, \overset{\nabla}{W}_{qs}(\omega) \;. \tag{5-78}$$

We now assume that $\varepsilon_{mn}(\omega)$ is of the same order of magnitude as the modal pressure, $\overset{\nabla}{P}_{mn}(\omega)$, and is therefore sufficiently small, in comparison with the externally applied modal force, $\overset{\nabla}{F}_{mn}(\omega)$, to be neglected. Under this assumption, it follows that the modal amplitudes, $\overset{\nabla}{W}_{mn}(\omega)$, can be approximated by

$$\overset{\nabla}{W}_{mn}(\omega) \approx \frac{\overset{\nabla}{F}_{mn}(\omega)}{\{Dk_{mn}^{4} - [\mu + \mu_{mnmn}(\omega)]\omega^{2} + i[r + r_{mnmn}(\omega)]\omega\}} \;. \tag{5-79}$$

It therefore follows that the wavevector-frequency description of the displacement field of the baffled, lightly fluid-loaded, simply supported plate can be approximated by

$$W(\underline{k},\omega) \approx \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\overset{\nabla}{F}_{mn}(\omega) I_{mn}(\underline{k})}{\{Dk_{mn}^{4} - [\mu + \mu_{mnmn}(\omega)]\omega^{2} + i[r + r_{mnmn}(\omega)]\omega\}} . \quad (5\text{-}80)$$

Comparison of equation (5-80) with equation (4-217) of section 4.3.3.2 shows that the wavevector-frequency description of the displacement field of the lightly fluid-loaded, simply supported plate has a mathematical form similar to that of the simply supported plate in vacuo. Indeed, this comparison reveals that the effect of the light fluid loading is to increase the apparent mass and damping of the plate. The additional mass and damping are modally dependent quantities, which, in this first order approximation, result only from the autocoupled ($m = q$ and $n = s$) modal contributions to the pressure field at the surface of the plate.

The similarity in the mathematical forms of equations (5-80) and (4-217) implies a similarity between the wavevector-frequency characteristics of in-vacuo and lightly fluid-loaded, simply supported plates. By arguments presented in section 4.3.3.2, certain wavevector-frequency characteristics of the forced response of the in-vacuo, simply supported plate were deduced from equation (4-217). By taking proper account of the differences in inertia and damping between equations (4-127) and (5-80), a similar set of wavevector-frequency characteristics can be inferred for the forced response of the lightly fluid-loaded, simply supported plate.

Recall that at any resonance frequency, ω_{MN}, the magnitude of the wavevector-frequency response of the simply supported plate in vacuo was argued to be relatively large in the neighborhoods of those modal wavevectors associated with the resonance (i.e., where $|\underline{k}| \approx k_{MN}$) and in the neighborhoods of those modal wavevectors associated with relatively large modal forces. By similarity arguments, it follows that, at any resonance frequency of the lightly fluid-loaded, simply supported plate, ω_{MN}', the magnitude of the wavevector-frequency response will be relatively large in the neighborhoods of those modal wavevectors associated with the resonance (i.e., where $|\underline{k}| \approx k_{MN}$) and in the neighborhoods of those modal wavevectors associated with relatively large modal forces. Here, ω_{MN}' is the natural frequency of the MN-th mode of the lightly fluid-loaded plate, which is

defined as that frequency at which

$$Dk_{MN}^{4} - [\mu + \mu_{MNMN}(\omega)]\omega^{2} = 0 . \tag{5-81}$$

In contrast, the MN-th modal natural frequency of the in-vacuo plate was defined as that frequency satisfying

$$Dk_{MN}^{4} - \mu\omega^{2} = 0 . \tag{5-82}$$

Thus, we see that lightly fluid-loaded and in-vacuo simply supported plates have similar wavevector-frequency characteristics at resonance. However, owing to the additional apparent mass associated with the fluid loading, the resonance frequencies of the lightly fluid-loaded plate differ from those of the plate in vacuo.

In section 4.3.3.2, we argued that, at a nonresonance frequency ω_0, the in-vacuo plate responded most strongly to those modal forces characterized by modal wavenumbers, k_{mn}, nearest to the free wavenumber, $k_p(\omega_0)$, of the in-vacuo plate: that is, those modal forces characterized by modal numbers, m and n, such that

$$k_{mn} = \sqrt{(m\pi/L_1)^2 + (n\pi/L_2)^2} \approx k_p(\omega_0) . \tag{5-83}$$

By similarity arguments, the lightly fluid-loaded plate also responds most strongly at a nonresonance frequency to modal forces characterized by modal wavenumbers, k_{mn}, nearest to the free wavenumber of the lightly fluid-loaded plate, $k_p(\omega_0)$, which (by equation (5-80)) can be approximated by

$$\tilde{k}_p(\omega_0) = \sqrt[4]{[\mu + \mu_{mnmn}(\omega_0)]\omega_0^{2}/D} . \tag{5-84}$$

Thus, off resonance, both the in-vacuo and lightly fluid-loaded plate respond most strongly to modal forces characterized by modal wavenumbers nearest to the free wavenumbers of the respective plates. However, owing to the additional mass associated with the fluid loading, the free wavenumber of the lightly fluid loaded plate differs from that of the plate in vacuo.

It was demonstrated in section 4.3.3.2 that excitation of the in-vacuo simply supported plate by a single wavevector-frequency component of the externally applied forcing field produced a response, at the frequency of excitation, that was comprised of a weighted distribution of wavevectors. This conversion of a single wavevector component of the input into multiple wavevector components of response was attributed to wavevector scattering at the boundaries of the plate. This mechanism for wavevector conversion is, of course, also present in the lightly fluid-loaded, simply supported plate. However, as was shown above, fluid loading provides an additional mechanism for wavevector conversion in the simply supported plate: that is, modal coupling. Whereas only autocoupling of modes is assumed in the light fluid-loading approximation, it should be recognized that the forces associated with crosscoupled modes can be significant for heavier fluid loadings.

5.4 CONCLUDING REMARKS

In this chapter, we have demonstrated that spatially distributed, multicomponent linear systems can be interpreted as an assemblage of coupled subsystems, where each subsystem represents a single physical component of the composite system and the couplings between subsystems reflect appropriate interactions between the corresponding physical components. By use of this interpretation, we argued that the mathematical model of a multicomponent system is formulated by appropriately coupling the assemblage of mathematical models of the subsystems associated with the composite system. To demonstrate this procedure for the formulation and solution of mathematical models of multicomponent systems, we treated the free and forced vibrations of an infinite plate with fluid loading on one side and the forced vibration of a baffled, simply supported plate with fluid loading on one side as illustrative examples.

These illustrative examples reveal that a primary effect of the fluid loading, common to both the infinite and simply supported plates, is to increase the apparent mass and damping of the plate. The increase in apparent mass, though frequency dependent, causes the free wavenumber of the fluid-loaded plate to be greater, at any frequency, than that of the same plate

in vacuo. Inasmuch as the free wavenumber defines the resonance characteristics of the plate in the wavevector-frequency domain, it follows that the fluid loading alters the resonance behavior of infinite and space-limited plates. In the case of the simply supported plate, it was shown that the fluid loading provides a means for wavevector conversion supplemental to the scattering mechanism associated with reflections at the boundaries of the in-vacuo plate. This additional means of wavevector conversion is modal coupling, whereby a single modal component of displacement of the plate produces a pressure field at the surface of the plate that is comprised of many modal components. These modal components of the pressure field act as additional modal forces on the plate and produce corresponding modal responses of the plate.

The response of multicomponent systems comprised of structural and fluid components is the focus of structural acoustics. The response of interest (i.e., output) in such systems can be either the vibratory field of the structure or the acoustic pressure field produced by the vibration of the structure. The illustrative examples for this chapter are a subset of perhaps the most exhaustively studied class of problems in structural acoustics: the vibration of, and radiation from, plates in contact with an acoustic fluid. Owing to space limitations and the desire to provide a simple and consistent set of illustrative examples of coupled systems, the examples presented here focus only on the vibratory displacement fields of the simplest space-invariant and space-limited forms of plate-fluid systems: that is, the infinite and simply supported fluid-loaded plates. Space limitations also restricted the detail to which the effects of fluid-loading were examined. To supplement the treatment of coupled plate-fluid systems provided here and to demonstrate the variety of plate-fluid systems addressed in the literature, we close this chapter with a brief listing of references. These references can be used as a springboard by the interested reader to further expand his sources of information.

The vibration and pressure fields associated with uniform, infinite plates under various forms of excitations constitute the most extensively studied class of coupled systems in structural acoustics. Examples of these systems are treated in such standard texts as Junger and Feit[6] and Morse and

Ingard.[7] However, as late as 1979, there was continuing interest[8,9,10] (and some confusion) regarding the free waves of fluid-loaded plates. The vibration and acoustic fields associated with fluid-loaded infinite plates are usually obtained by asymptotic methods. Examples of such approaches are presented by Morse and Ingard[11] and by Creighton.[12,13]

The vibration and acoustic fields of the simply supported, rectangular plate are treated by Junger and Feit.[14] However, for a detailed treatment of the modal coupling terms (i.e., μ_{mnqs} and r_{mnqs}) and high and low frequency approximate solutions of the displacement and radiated fields of fluid-loaded, simply supported plates, the reader is referred to Davies.[15] In addition, Maidanik[16] devised a method for classifying the various modes of simply supported plates in terms of their radiation efficiencies. Creighton and Innes[13] apply asymptotic methods to obtain approximate solutions for the vibration and radiation fields of certain other examples of fluid-loaded, space-limited plates.

Finally, the vibration and acoustic radiation fields of beam-stiffened, fluid-loaded plates have received much attention over the past 25 years. This work has progressed from the consideration of the vibration and acoustic fields of a single beam attached to a plate,[17,18] through the treatment of the vibration and pressure fields associated with periodically stiffened, fluid-loaded plates,[19] to the prediction of the vibratory and radiated fields of a plate with any number of supports.[20,21]

The above references illustrate the variety of coupled plate-fluid systems encountered in structural acoustics and provide a starting point for readers interested in such systems.

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